XXII. *Researches in physical astronomy.* *By* JOHN WILLIAM LUBBOCK, *Esq. Fellow of the Royal Society.*

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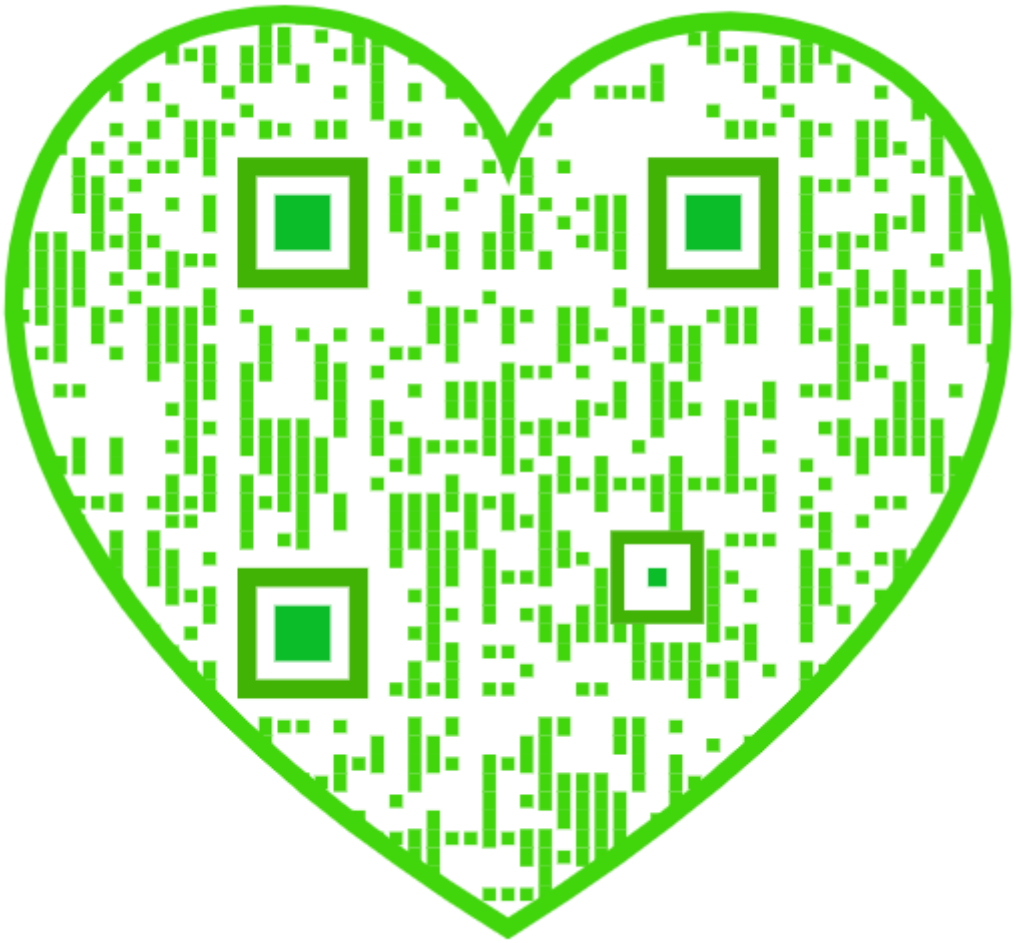
IN the first volume of the Mécanique Céleste, LAPLACE has given expressions for the variations of the elliptic constants, which are true when the square and higher powers of the disturbing force are neglected; and he has proved, upon the supposition that the planets move in the same direction, in orbits nearly circular and little inclined one to another, that the eccentricities and inclinations vary within small limits, thereby demonstrating within these conditions the stability of the planetary system. But these conditions are not necessary to the stability of a system of bodies, subject to the law of attraction, which obtains in our system. I have given in the following investigation the expressions for the variations of the elliptic constants, which are rigorously true whatever power of the disturbing force be retained; and it is easy to conclude from the form of their expressions, that however far the approximation be carried, the eccentricity, the major axis, and the tangent of the inclination of the orbit to a fixed plane, contain no term which varies with the time; their variations are all periodic, and they oscillate therefore within certain limits. This theorem is no longer true if the planet moves in a resisting medium.

I have also given some equations which obtain when an angle is taken for the independent variable, which in the elliptic movement is the eccentric anomaly, which are of remarkable simplicity, and which, as far as I know, have never been noticed, and the development of the disturbing function R to the quantities involving the squares and products of the eccentricities inclusive.



Let x, y, z denote the rectangular co-ordinates

r ..... distance from the sun

r' ..... distance from the sun projected upon the plane x, y

*λ ..... longitude reckoned upon the plane of its orbit

λ' ..... longitude reckoned upon the plane $x\,y$

s ..... tangent of the latitude

v .... a variable, which in the elliptic theory is the eccentric anomaly

m .... the mass

a ..... semiaxis major

e ..... eccentricity

ϖ ..... longitude of the perihelion

ε ..... longitude of the epoch

ν ..... longitude of the ascending node

ι ..... inclination of the orbit to the plane $x\,y$

α .... a constant quantity which accompanies v

} of the planet m.

M .... the mass of the sun.

$$M + m = \mu, \quad \sqrt{\frac{\mu}{a^3}} = n.$$

$x = r'\cos\lambda'$, $y = r'\sin\lambda'$, $z = r's$, and in the elliptic motion $r'^2\mathrm{d}\lambda' = h\,\mathrm{d}\,t$.

$$R = m_{,}\left\{\frac{x x_{,} + y y_{,} + z z_{,}}{\{x_{,}^2 + y_{,}^2 + z_{,}^2\}^{\frac{3}{2}}} - \frac{1}{\{(x-x_{,})^2 + (y-y_{,})^2 + (z-z_{,})^2\}^{\frac{1}{2}}}\right\}$$

$$= m_{,}\left\{\frac{r'\{\cos(\lambda'-\lambda'_{,}) + s s_{,}\}}{r'^2_{,}(1+s_{,}^2)^{\frac{3}{2}}} - \frac{1}{\left\{r'^2(1+s^2) - 2r'r'_{,}\{\cos(\lambda'-\lambda'_{,}) + s s_{,}\} + r'^2_{,}(1+s_{,}^2)\right\}^{\frac{1}{2}}}\right\}$$

* Laplace uses the letter v to denote longitude, u the eccentric anomaly, and φ the inclination of the orbit to a fixed plane; but as v is very frequently used to signify velocity, and φ geographical latitude, and as the letters of the Greek alphabet are generally used for angles, I have taken the letters λ, υ, and ι for these quantities.

$$= -m_{,}\left\{\begin{array}{l}\dfrac{1}{r_{,}'\sqrt{1+s_{,}^{2}}}+\dfrac{r'^{2}}{4r_{,}'^{3}}\{1+3\cos(2\lambda'-2\lambda_{,}')+12\,s\,s_{,}\cos(\lambda'-\lambda_{,}')-2\,s^{2}\}\\ \qquad+\dfrac{r'^{3}}{8r_{,}'^{4}}\{3(1-4\,s^{2})\cos(\lambda'-\lambda_{,}')+5\cos(3\lambda'-3\lambda_{,}')\}\\ \qquad+\dfrac{r'^{4}}{64r_{,}'^{5}}\{9+20\cos(2\lambda'-2\lambda_{,}')+35\cos(4\lambda'-4\lambda_{,}')\}\end{array}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)=m_{,}\left\{\frac{\cos(\lambda'-\lambda_{,}')+s\,s_{,}}{r_{,}'^{2}(1+s_{,}^{2})^{\frac{3}{2}}}+\frac{r'(1+s^{2})-r_{,}'\{\cos(\lambda'-\lambda_{,}')+s\,s_{,}\}}{\{r'^{2}(1+s^{2})-2\,r'r_{,}'\{\cos(\lambda'-\lambda_{,}')+s\,s_{,}\}+r_{,}'^{2}(1+s_{,}^{2})\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)=-m_{,}\left\{\frac{r'\sin\lambda'-\lambda_{,}'}{r_{,}'^{2}(1+s'^{2})^{\frac{3}{2}}}-\frac{r'r_{,}'\sin(\lambda'-\lambda_{,}')}{\left\{r'^{2}(1+s^{2})-2\,r'r_{,}'\{\cos(\lambda'-\lambda_{,}')+s\,s_{,}\}+r_{,}'^{2}(1+s_{,}^{2})\right\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)=m_{,}\left\{\frac{r's_{,}}{r_{,}'^{2}(1+s_{,}^{2})^{\frac{3}{2}}}+\frac{r'^{2}s-r'r_{,}'s_{,}}{\left\{r'^{2}(1+s^{2})-2\,r'r_{,}'\{\cos(\lambda'-\lambda_{,}')+s\,s_{,}\}+r_{,}'^{2}(1+s_{,}^{2})\right\}^{\frac{3}{2}}}\right\}$$

Let P be the place of the planet m, S the place of the sun, S N the intersection of the orbits of m and $m_{,}$, S L the line from which longitudes are reckoned, $\mathrm{P}_{,}'$, the projection of $\mathrm{P}_{,}$ upon the plane of the orbit of P; then if the plane $x\,y$ coincide with the orbit of m, $\mathrm{S\,P}=r$, $\mathrm{P\,S\,L}=\lambda$, $\mathrm{N\,S\,L}=\nu$, $\mathrm{P}_{,}\mathrm{S\,N}+\mathrm{N\,S\,L}=\lambda_{,}$, $\mathrm{P}_{,}'\mathrm{S\,L}=\lambda_{,}'$, $\mathrm{S\,P}_{,}'=r_{,}'$, $\mathrm{S\,P}_{,}=r_{,}=r_{,}'\,(1+s_{,}^{2})^{\frac{1}{2}}$

$$R=m_{,}\left\{\frac{\mathrm{S\,P}\times\mathrm{S\,P}_{,}\cos\mathrm{P\,S\,P}_{,}}{\mathrm{S\,P}_{,}^{3}}-\frac{1}{\{\mathrm{S\,P}^{2}-2\,\mathrm{S\,P}\times\mathrm{S\,P}_{,}\cos\mathrm{P\,S\,P}_{,}+\mathrm{S\,P}_{,}^{2}\}^{\frac{1}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=m_{,}\left\{\frac{\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'}{\mathrm{S\,P}_{,}^{3}}+\frac{\mathrm{S\,P}-\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'}{\{\mathrm{S\,P}^{2}-2\,\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'+\mathrm{S\,P}_{,}^{2}\}^{\frac{3}{2}}}\right\}$$

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=m_{,}\left\{\frac{\mathrm{S\,P}\times\mathrm{S\,P}_{,}\cos\mathrm{P\,S\,P}_{,}}{\mathrm{S\,P}_{,}^{3}}+\frac{\mathrm{S\,P}^{2}-\mathrm{S\,P}\times\mathrm{S\,P}_{,}\cos\mathrm{P\,S\,P}_{,}}{\{\mathrm{S\,P}^{2}-2\,\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'+\mathrm{S\,P}_{,}^{2}\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)=-m_{,}\left\{\frac{\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\sin\mathrm{P\,S\,P}_{,}'}{\mathrm{S\,P}_{,}^{3}}-\frac{\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\sin\mathrm{P\,S\,P}_{,}'}{\{\mathrm{S\,P}^{2}-2\,\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'+\mathrm{S\,P}_{,}^{2}\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)=m_{,}\left\{\frac{\mathrm{S\,P}\times\mathrm{P}_{,}\mathrm{P}_{,}'}{\mathrm{S\,P}_{,}^{3}}-\frac{\mathrm{S\,P}\times\mathrm{P}_{,}\mathrm{P}_{,}'}{\{\mathrm{S\,P}^{2}-2\,\mathrm{S\,P}\times\mathrm{S\,P}_{,}'\cos\mathrm{P\,S\,P}_{,}'+\mathrm{S\,P}_{,}^{2}\}^{\frac{3}{2}}}\right\}$$

$$\cos(\lambda-\lambda_{,})=\cos(\lambda-\nu)\cos(\lambda_{,}-\nu)+\cos\imath\sin(\lambda-\nu)\sin(\lambda_{,}-\nu)$$

$$=\cos^{2}\frac{\imath}{2}\cos(\lambda-\lambda_{,})+\sin^{2}\frac{\imath}{2}\cos(\lambda+\lambda_{,}-2\,\nu)$$

$$\tan(\lambda_{,}'-\nu)=\cos\imath\tan(\lambda_{,}-\nu)$$

$$\sin(\lambda_{,}'-\nu)=\frac{\cos\imath\tan(\lambda_{,}-\nu)}{(1+\cos^{2}\imath\tan^{2}(\lambda_{,}-\nu))^{\frac{1}{2}}}$$

$$\cos(\lambda_l' - \nu) = \frac{1}{(1 + \cos^2 \iota \tan^2(\lambda_l - \nu))^{\frac{1}{2}}}$$

$$r r_l' \sin(\lambda - \lambda_l') = r r_l' \{\sin\lambda \cos\lambda_l' - \cos\lambda \sin\lambda_l\}$$

$$\cos\lambda_l' = \cos(\lambda_l' - \nu)\cos\nu - \sin(\lambda_l' - \nu)\sin\nu$$

$$= \frac{\cos(\lambda_l - \nu)\cos\nu - \sin(\lambda_l - \nu)\sin\nu\left(1 - 2\sin^2\frac{\iota}{2}\right)}{(1 - \sin^2\iota \sin^2(\lambda_l - \nu))^{\frac{1}{2}}}$$

$$= \frac{\cos\lambda_l + 2\sin^2\frac{\iota}{2}\sin(\lambda_l - \nu)\sin\nu}{(1 - \sin^2\iota \sin^2(\lambda_l - \nu))^{\frac{1}{2}}}$$

$$\sin\lambda_l' = \sin(\lambda_l' - \nu)\cos\nu + \cos(\lambda_l' - \nu)\sin\nu$$

$$= \frac{\sin(\lambda_l - \nu)\cos\nu\left(1 - 2\sin^2\frac{\iota}{2}\right) + \cos(\lambda_l - \nu)\sin\nu}{(1 - \sin^2\iota \sin^2(\lambda_l - \nu))^{\frac{1}{2}}}$$

$\sin \mathrm{P}_l \mathrm{S}\, \mathrm{P}_l' = \sin\iota \sin(\lambda_l - \nu)$, $r_l' = r_l \cos \mathrm{P}_l \mathrm{S}\, \mathrm{P}_l' = r_l\left(1 - \sin^2\iota \sin^2(\lambda_l - \nu)\right)^{\frac{1}{2}}$ therefore,

$$r r_l' \sin(\lambda - \lambda_l') = r r_l \left\{\sin(\lambda - \lambda_l) + 2\sin^2\frac{\iota}{2}\sin(\lambda_l - \nu)\cos(\lambda - \nu)\right\}$$

$$= r r_l \left\{\cos^2\frac{\iota}{2}\sin(\lambda - \lambda_l) + \sin^2\frac{\iota}{2}\sin(\lambda + \lambda_l - 2\nu)\right\}$$

similarly it may be shown that

$$r r_l' \cos(\lambda - \lambda_l') = r r_l \left\{\cos^2\frac{\iota}{2}\cos(\lambda - \lambda_l) + \sin^2\frac{\iota}{2}\cos(\lambda + \lambda_l - 2\nu)\right\}$$

$$\mathrm{S}\, \mathrm{P}_l = \frac{a_l(1 - e_l^2)}{1 + e_l \cos(\mathrm{P}_l \mathrm{S}\, \mathrm{N} + \mathrm{N}\, \mathrm{S}\, \mathrm{L} - \varpi_l)} = \frac{a_l(1 - e_l^2)}{1 + e_l \cos(\lambda_l - \varpi_l)}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mu x}{r^3} + \left(\frac{\mathrm{d}R}{\mathrm{d}x}\right) = 0, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mu y}{r^3} + \left(\frac{\mathrm{d}R}{\mathrm{d}y}\right) = 0, \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mu z}{r^3} + \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right) = 0$$

$$\frac{r'^2 \mathrm{d}\lambda'^2 + \mathrm{d}r'^2(1 + s^2) + 2r' s\, \mathrm{d}r'\, \mathrm{d}s + r'^2 \mathrm{d}s^2}{\mathrm{d}t^2} - \frac{2\mu}{r'(1 + s^2)^{\frac{1}{2}}} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0$$

$\mathrm{d}R$ being the differential of R with regard only to the co-ordinates of the planet m.

$$\frac{\mathrm{d}^2 r'(1 + s^2) - r'\mathrm{d}\lambda'^2 + 2s\, \mathrm{d}r'\, \mathrm{d}s + r' s\, \mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{\mu}{r'(1 + s^2)^{\frac{1}{2}}} + \left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) = 0$$

$$\mathrm{d}.\frac{r'^2 \mathrm{d}\lambda'}{\mathrm{d}t} + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right) = 0$$

$$\frac{r' s\, \mathrm{d}^2 r' + 2r'\, \mathrm{d}r'\, \mathrm{d}s + r'^2 \mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{s}{r'(1 + s^2)^{\frac{1}{2}}} + \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$\frac{r' d^2 r' - r'^2 d\lambda'^2}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) = 0$$

$$\frac{r'^2 d^2 s + 2 r' dr' ds - r'^2 s d\lambda'^2}{dt^2} + (1+s^2)\left(\frac{dR}{ds}\right) - r' s\left(\frac{dR}{dr'}\right) = 0$$

$$\frac{d^2 . r'^2 (1+s^2)}{2 dt^2} - \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + \frac{\mu}{a} + 2\int dR + r'\left(\frac{dR}{dr'}\right) = 0$$

Making λ the independent variable instead of t,

$$\frac{d^2 r'}{dt^2} - 2\frac{dr' d^2 t}{dt^3} - r'^2 \frac{d\lambda'^2}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) = 0$$

$$r'^4 \frac{d\lambda'^2}{dt^2} = h^2 - 2\int r'^2 \left(\frac{dR}{d\lambda'}\right) d\lambda', \; h \text{ being a constant,}$$

$$4 r'^3 dr' \frac{d\lambda'^2}{dt^2} - 2 r'^4 \frac{d\lambda'^2 d^2 t}{dt^3} = -2 r'^2 \left(\frac{dR}{d\lambda'}\right) d\lambda'$$

$$\left\{\frac{d^2 . \frac{1}{r'}}{d\lambda'^2} + \frac{1}{r'}\right\}\left\{1 - \frac{2}{h^2}\int r'^2 \left(\frac{dR}{d\lambda'}\right) d\lambda'\right\} - \frac{\mu}{h^2 (1+s^2)^{\frac{3}{2}}}$$

$$- \frac{r'}{h^2}\left\{r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) - \frac{1}{r'}\left(\frac{dR}{d\lambda'}\right)\frac{dr'}{d\lambda'}\right\} = 0$$

$$\left\{\frac{d^2 s}{d\lambda'^2} + s\right\}\left\{1 - \frac{2}{h^2}\int r'^2 \left(\frac{dR}{d\lambda'}\right) d\lambda'\right\}$$

$$+ \frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{dR}{ds}\right) - r'\left(\frac{dR}{dr'}\right) - \left(\frac{dR}{d\lambda'}\right)\frac{ds}{d\lambda'}\right\} = 0$$

When the disturbing force is neglected

$$\frac{d . r'^2 d\lambda'}{dt'} = 0, \quad \frac{d^2 . \frac{1}{r'}}{d\lambda'^2} + \frac{1}{r'} - \frac{\mu}{h^2 (1+s^2)^{\frac{3}{2}}} = 0, \quad \frac{d^2 s}{d\lambda'^2} + s = 0$$

of which equations the integrals are,

$$r'^2 d\lambda' = h\, dt, \quad \frac{1}{r'} = \frac{\mu \cos \iota^2}{h^2}\left\{(1+s^2)^{\frac{1}{2}} + e \cos(\lambda' - \varpi)\right\}$$

$$s = \tan \iota \sin(\lambda' - \nu).$$

If $dt = \sqrt{\frac{a}{\mu}}\, r'\, d\upsilon$, and υ be taken for the independent variable

$$r'\frac{d^2 r'}{dt^2} - r'\, dr' \frac{d^2 t}{dt^3} - r'^2 \frac{d\lambda'^2}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) = 0$$

$$\mathrm{d}^2 t = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \mathrm{d}r' \,\mathrm{d}\upsilon$$

$$\frac{\mathrm{d}^2 r'}{\mathrm{d}\upsilon^2} + \left(\frac{\mathrm{d}.r's}{\mathrm{d}\upsilon}\right)^2 - \frac{a}{(1+s^2)^{\frac{1}{2}}} + r' + \frac{ar'}{\mu}\left\{2\int \mathrm{d}R + r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\right\} = 0.$$

If the orbit of the planet m coincide with the plane $x\,y$, s is of the order of the disturbing force, of which therefore neglecting the square, $r' = r$, $\lambda' = \lambda$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\upsilon^2} - a + r + \frac{ar}{\mu}\left\{2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\} = 0.$$

When the disturbing force is neglected, the integral of this equation is $r = a\{1 - e\cos(\upsilon - \alpha)\}$, υ being the eccentric anomaly.

$$n\,t + \varepsilon - \varpi = \upsilon - \alpha - e\sin(\upsilon - \alpha)$$

$$\tan\frac{\lambda - \varpi}{2} = \left\{\frac{1+e}{1-e}\right\}^{\frac{1}{2}} \tan\frac{\upsilon - \alpha}{2}.$$

If Q be put for the quantity $\frac{ar}{\mu}\left\{2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}$ and the constant α, which may afterwards be replaced, be omitted for the present,

$$r = a\{1 - e\cos\upsilon\} - \sin\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon + \cos\upsilon\int Q\sin\upsilon\,\mathrm{d}\upsilon$$

$$\mathrm{d}t = \sqrt{\frac{a}{\mu}}\left\{a\{1 - e\cos\upsilon\} - \sin\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon + \cos\upsilon\int Q\sin\upsilon\,\mathrm{d}\upsilon\right\}\mathrm{d}\upsilon$$

$$n\,t + \varepsilon - \varpi = \upsilon - e\sin\upsilon - \frac{1}{a}\left\{\int Q\,\mathrm{d}\upsilon - \cos\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon - \sin\upsilon\int Q\sin\upsilon\,\mathrm{d}\upsilon\right\}$$

If $\upsilon = \mathrm{f}(n\,t + \varepsilon - \varpi)$ in the elliptic theory, then neglecting the square of the disturbing force,

$$\upsilon = \mathrm{f}(n\,t + \varepsilon - \varpi) + \frac{\mathrm{d}.\mathrm{f}(nt + \varepsilon - \varpi)}{\mathrm{d}t}\frac{1}{a}\left\{\int Q\,\mathrm{d}\upsilon - \cos\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon - \sin\upsilon\int Q\sin\upsilon\right\}\mathrm{d}\upsilon$$

If $\delta\upsilon$, δr denote the values of those parts of r and υ which are due to the first power of the disturbing force,

$$\delta\upsilon = \frac{\mathrm{d}\upsilon}{a\,n\,\mathrm{d}t}\left\{\int Q\,\mathrm{d}\upsilon - \cos\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon - \sin\upsilon\int Q\sin\upsilon\,\mathrm{d}\upsilon\right\}$$

$$\delta r = a\,e\sin\upsilon\,\delta\upsilon - \sin\upsilon\int Q\cos\upsilon\,\mathrm{d}\upsilon + \cos\upsilon\int Q\sin\upsilon\,\mathrm{d}\upsilon$$

$$= \frac{\sin\upsilon}{1 - e\cos\upsilon}\int Q\{e - \cos\upsilon\}\,\mathrm{d}\upsilon - \frac{e - \cos\upsilon}{1 - e\cos\upsilon}\int Q\sin\upsilon\,\mathrm{d}\upsilon.$$

In the elliptic theory,

$$\mathrm{d}\upsilon = \frac{n\,\mathrm{d}t}{1 - e\cos\upsilon}, \qquad \frac{\cos\upsilon - e}{1 - e\cos\upsilon} = \cos\lambda, \qquad \frac{\sin\upsilon(1 - e^2)^{\frac{1}{2}}}{1 - e\cos\upsilon} = \sin\lambda$$

therefore,

$$\delta r = \frac{an\cos\lambda \int r\sin\lambda \left\{2\int dR + r\left(\frac{dR}{dr}\right)\right\} dt - an\sin\lambda \int r\cos\lambda \left\{2\int dR + r\left(\frac{dR}{dr}\right)\right\} dt}{\mu(1-e^2)^{\frac{1}{2}}}$$

which is the equation X of the Mécanique Céleste, vol. i. p. 258.

Multiplying the equation of p. 114, l. 2, by $\frac{2r'^2 d\lambda'}{dt}$, and integrating, $\frac{r'^4 d\lambda'^2}{dt^2}$ $+2\int r'^2\left(\frac{dR}{d\lambda'}\right)d\lambda' = h_0^2$, if $h^2 = h_0^2 - 2\int r'^2\left(\frac{dR}{d\lambda'}\right)d\lambda'$, h being variable, and h_0 the value of h at a given epoch, $h\,dh = -r'^2\left(\frac{dR}{d\lambda'}\right)d\lambda'$, $r^2 d\lambda' = h\,dt$, and making λ' the independent variable instead of t, $2r'\,dr'\,d\lambda' = dh\,dt + h\,d^2t$

$$r'\frac{d^2r'}{dt^2} - \frac{r'dr'd^2t}{dt^3} - \frac{r'^2d\lambda'^2}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{3}{2}}} + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) = 0$$

$$r'\left(\frac{d^2r'}{dt^2}\right) + \frac{r'dr'dt}{h\,dt^2} - \frac{2r'^2dr'^2d\lambda}{h\,dt^3} - \frac{r'^2d\lambda'^2}{dt^2} + \frac{\mu}{r'(1+s^2)^{\frac{3}{2}}} + r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) = 0$$

$$d.\frac{1}{\frac{r'}{d\lambda'^2}} + \frac{1}{r'} - \frac{\mu}{h^2(1+s^2)^{\frac{3}{2}}} - \frac{r'}{h^2}\left\{r'\left(\frac{dR}{dr'}\right) - s\left(\frac{dR}{ds}\right) - \frac{1}{r'}\left(\frac{dR}{d\lambda'}\right)\frac{dr'}{d\lambda'}\right\} = 0$$

$$\frac{d^2s}{d\lambda'^2} + s + \frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{dR}{ds}\right) - s\left(\frac{dR}{ds}\right) - \left(\frac{dR}{d\lambda'}\right)\frac{ds'}{d\lambda'}\right\} = 0$$

If all the constants in the elliptic integrals are supposed to vary, subject to the condition that they still satisfy these differential equations, and that the form of the first differential coefficients $\frac{dr'}{d\lambda'}$, $\frac{ds}{d\lambda'}$, remains unaltered,

$$d.\frac{1}{\frac{r'}{d\lambda'}} = \frac{\mu\cos\iota^2}{h^2}\left\{\frac{s}{(1+s^2)^{\frac{1}{2}}}\frac{ds}{d\lambda'} - e\sin(\lambda'-\varpi)\right\}, \qquad \frac{ds}{d\lambda'} = \tan\iota\cos(\lambda'-\nu)$$

$$-2\{(1+s^2)^{\frac{1}{2}} + e\cos(\lambda'-\varpi)\}\{h^2\sin\iota\,d\iota + \cos\iota\,h\,dh\}$$
$$+h^2\cos\iota\cos(\lambda'-\varpi)\,de + h^2\cos\iota\,e\sin(\lambda'-\varpi)\,d\varpi = 0$$

$$-2\left\{\frac{s\tan\iota\cos(\lambda'-\nu)}{(1+s^2)^{\frac{1}{2}}} - e\sin(\lambda'-\varpi)\right\}\{h^2\sin\iota\,d\iota + \cos\iota\,h\,dh\}$$
$$-h^2\cos\iota\sin(\lambda'-\varpi)\,de + h^2\cos\iota\,e\cos(\lambda'-\varpi)\,d\varpi$$
$$-\frac{\cos\iota\,sr'^2}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{dR}{ds}\right) - r'\left(\frac{dR}{dr'}\right) - \left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\}d\lambda'$$

$$-\frac{h^2 r'}{\mu \cos \iota}\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-\frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

$$\sin(\lambda'-\nu)\frac{\mathrm{d}\iota}{\cos\iota^2}-\tan\iota\cos(\lambda'-\nu)\,\mathrm{d}\nu=0$$

$$\cos(\lambda'-\nu)\frac{\mathrm{d}\iota}{\cos\iota^2}+\tan\iota\sin(\lambda'-\nu)\,\mathrm{d}\nu+\frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

$$h\,\mathrm{d}\,h=-r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$$

Whence by elimination,

$$h^2\cos\iota\,\mathrm{d}\,e+2\left\{(1+s^2)^{\frac{1}{2}}\cos(\lambda'-\varpi)+e-\frac{\tan\iota\, s\cos(\lambda'-\nu)\sin(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\right\}$$

$$\left\{\sin\iota\cos^2\iota\, r'^2\cos(\lambda'-\nu)\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}+\cos\iota\, r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'$$

$$+\frac{s\cos\iota\, r'^2\sin(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'$$

$$+\frac{h^2 r'}{\mu\cos\iota}\sin(\lambda'-\varpi)\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-\frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

$$h^2\cos\iota\, e\,\mathrm{d}\,\varpi+2\left\{(1+s^2)^{\frac{1}{2}}\sin(\lambda'-\varpi)+\frac{s\tan\iota\cos(\lambda'-\nu)\cos(\lambda'-\varpi)}{(1+s^2)^{\frac{1}{2}}}\right\}$$

$$\left\{\sin\iota\cos^2\iota\, r'^2\cos(\lambda'-\nu)\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}+\cos\iota\, r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'$$

$$-\frac{s\cos\iota\, r'^2\cos(\lambda-\varpi)}{(1+s^2)^{\frac{1}{2}}}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'$$

$$-\frac{h^2 r'}{\mu\cos\iota}\cos(\lambda'-\varpi)\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-\frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

$$d\iota+\frac{r'^2\cos\iota^2\cos(\lambda'-\nu)}{h^2}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

$$d\nu+\frac{r'^2\sin(\lambda'-\nu)}{h^2\tan\iota}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-s\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda'=0$$

If $x = \frac{r \cos \lambda'}{(1+s^2)^{\frac{1}{2}}}$, $y = \frac{r \sin \lambda'}{(1+s^2)^{\frac{1}{2}}}$, $z = \frac{r s}{(1+s^2)^{\frac{1}{2}}}$,

$$\frac{dr^2 + \frac{r^2}{1+s^2}\left(\frac{ds^2}{1+s^2} + d\lambda'^2\right)}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int dR = 0$$

$$\frac{d^2 r}{dt^2} - \frac{r}{1+s^2}\frac{\left(\frac{ds^2}{1+s^2} + d\lambda'^2\right)}{dt^2} + \frac{\mu}{r^2} + \left(\frac{dR}{dr}\right) = 0$$

$$d \cdot \frac{\frac{r^2}{1+s^2} \cdot \frac{d\lambda'}{dt}}{dt} + \left(\frac{dR}{d\lambda'}\right) = 0$$

$$\frac{r^2 d^2 s + 2 r dr ds - \frac{2 r^2 s ds^2}{(1+s^2)} + r^2 s d\lambda'^2}{dt^2} + (1+s^2)^2 \left(\frac{dR}{ds}\right) = 0$$

Of which equations the integrals are

$$\frac{r^2}{1+s^2} \cdot d\lambda' = h\, dt, \quad \frac{1}{r} = \frac{\mu \cos i^2}{h^2 (1+s^2)^{\frac{1}{2}}} \left\{ (1+s^2)^{\frac{1}{2}} + e \cos (\lambda' - \varpi) \right\}$$

$$s = \tan i \sin (\lambda' - \nu)$$

If $dt = \sqrt{\frac{a}{\mu}}\, r\, d\upsilon$, and υ be taken for the independent variable

$$\frac{d^2 r}{d\upsilon^2} - a + r + \frac{ar}{\mu}\left\{ 2\int dR + r\left(\frac{dR}{dr}\right)\right\} = 0$$

$r = a\{1 - e' \cos(\upsilon - \alpha)\}$ in the elliptic motion.

e' is accented for the present in order to distinguish it from e.

If the constants in the elliptic integrals are supposed to vary, subject to the condition that they still satisfy these differential equations, and that the form of the first differential coefficient $\frac{dr}{d\upsilon}$ remains unaltered,

$$d^2 t = \frac{r\, d\upsilon\, da}{2\sqrt{a\mu}} + \sqrt{\frac{a}{\mu}}\, dr\, d\upsilon, \quad \frac{r d^2 r}{dt^2} = \frac{r dr^2}{dt^3} - \frac{r dr d^2 t}{dt^3}$$

$$\frac{d^2 r}{d\upsilon^2} - a + r + \frac{ar^3}{\mu}\left(\frac{dR}{dr}\right) - \frac{dr\, da}{2a\, d\upsilon^2} = 0$$

$$\left(1 - e' \cos(\upsilon - \alpha)\right) da - a \cos(\upsilon - \alpha)\, de' - a e' \sin(\upsilon - \alpha)\, d\alpha = 0$$

$$e' \sin(\upsilon - \alpha)\, da + a \sin(\upsilon - \alpha)\, de' - a e' \cos(\upsilon - \alpha)\, d\alpha$$

$$+\frac{ar^2}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v+\frac{a^2e}{\mu}\sin(v-\alpha)\,\mathrm{d}R=0$$

$$\mathrm{d}a=-\frac{2a^2}{\mu}\mathrm{d}R$$

$$\mathrm{d}e'=\frac{a}{\mu}\{2e'-2\cos(v-\alpha)-e'\sin(v-\alpha)^2\}\,\mathrm{d}R-\frac{r^2}{\mu}\sin(v-\alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v$$

$$e\,\mathrm{d}\alpha=\frac{a}{\mu}\sin(v-\alpha)\{e'\cos(v-\alpha)-2\}\,\mathrm{d}R+\frac{r^2}{\mu}\cos(v-\alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}v$$

$$\int n\,\mathrm{d}t+\varepsilon-\varpi=v-\alpha-e'\sin(v-\alpha)$$

$$\mathrm{d}\varepsilon-\mathrm{d}\varpi=-\mathrm{d}\alpha-\sin(v-\alpha)\,\mathrm{d}e'+e'\cos(v-\alpha)\,\mathrm{d}\alpha.$$

The equation of condition which obtains between the constants a, e, ϖ, ν, ι and h may be found from the equation

$$\frac{\mathrm{d}r^2+\frac{r^2}{1+s^2}\left(\frac{\mathrm{d}s^2}{1+s^2}+\mathrm{d}\lambda^2\right)}{\mathrm{d}t^2}-\frac{2\mu}{r}+\frac{\mu}{a}=0$$

$$\frac{\mathrm{d}r^2}{\mathrm{d}t^2}+\frac{h^2}{r^2\cos\iota^2}-\frac{2\mu}{r}+\frac{\mu}{a}=0$$

which gives

$$\frac{\mu\cos\iota^2}{h^2}\left\{e^2\cos^2\iota\sin^2(\nu-\varpi)-\{1+e\cos(\nu-\varpi)\}\{1-e\cos(\nu-\varpi)\}\right\}+\frac{1}{a}=0$$

Equating the values of r which have been found,

$$\frac{h^2\sqrt{1+s^2}}{\mu\cos\iota^2\{\sqrt{1+s^2}+e\cos(\lambda-\varpi)\}}=a\{1-e'\cos(v-\alpha)\}$$

since the origin of the angle v is arbitrary, we may suppose $\lambda-\varpi=0$, and $v-\alpha=0$ at the same time,

so that $$\frac{h^2\sqrt{1+\tan\iota^2\sin(\varpi-\nu)^2}}{\mu\cos\iota^2\{\sqrt{1+\tan\iota^2\sin(\varpi-\nu)^2}+e\}}=a(1-e')$$

All the equations which have hitherto been proved are rigorously true, whatever powers of the disturbing function be retained. They are susceptible of simplification when the square and higher powers of the disturbing function are neglected: in this case, if the orbit be supposed to coincide with the plane $x\,y$, $\tan\iota=0$, and if the longitude be reckoned from the perihelion of the planet P,

$$h^2\,\mathrm{d}e+\{2\cos\lambda+e+e\cos\lambda^2\}r^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda+\frac{h^2r^2\sin\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\lambda=0$$

$$h^2 e \, \mathrm{d}\varpi + \{2 + \cos\lambda\} \sin\lambda \, r^2 \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) \mathrm{d}\lambda - h^2 r^2 \cos\lambda \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) \mathrm{d}\lambda = 0$$

$$\mathrm{d}\iota + r^2 \frac{\cos(\lambda - \nu)}{\mu} \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) \mathrm{d}\lambda = 0$$

$$\mathrm{d}\nu + r^2 \frac{\sin(\lambda - \nu)}{\mu} \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) \mathrm{d}\lambda = 0$$

by equation, p. 336, line 12, $\mu\, a\,(1 - e^2) = h^2$, and by equation, line 17, $e = e'$.

$$\mathrm{d}e = \frac{a}{\mu}\left\{2e - 2\cos(\upsilon - \alpha) - e\sin(\upsilon - \alpha)^2\right\} \mathrm{d}R - \frac{r^2}{\mu}\sin(\upsilon - \alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) \mathrm{d}\upsilon$$

$$e\,\mathrm{d}\alpha = \frac{a}{\mu}\left\{e\cos(\upsilon - \alpha) - 2\right\}\sin(\upsilon - \alpha)\,\mathrm{d}R + \frac{r^2}{\mu}\cos(\upsilon - \alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\upsilon$$

$$\int n\,\mathrm{d}t + \varepsilon - \varpi = \upsilon - \alpha - e\sin(\upsilon - \alpha)$$

$$\mathrm{d}\varepsilon - \mathrm{d}\varpi = -\{1 - e\cos(\upsilon - \alpha)\}\,\mathrm{d}\alpha - \sin(\upsilon - \alpha)\,\mathrm{d}e$$

$$\mathrm{d}\varepsilon - \mathrm{d}\varpi = -\frac{r^2}{a^2\sqrt{1 - e^2}}\,\mathrm{d}\varpi - \left\{\frac{r}{a(1 - e^2)} + 1\right\}\sin(\upsilon - \alpha)\,\mathrm{d}e$$

$$\frac{\frac{h^2}{\mu}}{1 + e\cos(\lambda - \varpi)} = a\{1 - e\cos(\upsilon - \alpha)\}$$

$$\mathrm{d}e = -\frac{a\,n\,\mathrm{d}t}{\mu\sqrt{1 - e^2}}\left\{2\cos(\lambda - \varpi) + e + e\cos(\lambda - \varpi)^2\right\}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{a^2 n\,\mathrm{d}t\sqrt{1 - e^2}}{\mu}\sin(\lambda - \varpi)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$$

$$e\,\mathrm{d}\varpi = -\frac{a\,n\,\mathrm{d}t}{\mu\sqrt{1 - e^2}}\left\{2 + e\cos(\lambda - \varpi)\right\}\sin(\lambda - \varpi)\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) + \frac{a^2 n\,\mathrm{d}t\sqrt{1 - e^2}}{\mu}\cos(\lambda - \varpi)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$$

$$\mathrm{d}\varepsilon - \mathrm{d}\varpi = -\sqrt{1 - e^2}\,\mathrm{d}\varpi + \frac{2a^2 n(1 - e^2)\,\mathrm{d}t}{\mu\{1 + e\cos(\lambda - \varpi)\}}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$$

If the longitudes be reckoned from the perihelion of the planet P,

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{a\,n(\cos\lambda + e)}{\mu\sqrt{1 - e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{m_1 a^2 n\sqrt{1 - e^2}}{\mu}\left\{\frac{r_1'\sin\lambda_1'}{r_1'^3} + \frac{r\sin\lambda - r_1'\sin\lambda_1'}{\{r^2 - 2rr_1'\cos(\lambda - \lambda_1') + r_1'^2\}^{\frac{3}{2}}}\right.$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = -\frac{a n \sin\lambda}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{m_l a^2 n\sqrt{1-e^2}}{\mu}\left\{\frac{r_l'\cos\lambda_l'}{r_l'^3} + \frac{r\cos\lambda - r_l'\cos\lambda_l'}{\{r^2 - 2rr_l'\cos(\lambda-\lambda_l') + r_l'^2\}^{\frac{3}{2}}}\right\}$$

and since $n\,\mathrm{d}t = \frac{r}{a}\,\mathrm{d}v$,

$$\frac{\mathrm{d}e}{\mathrm{d}v} = -\frac{r(\cos\lambda + e)}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) - \frac{m_l a r\sqrt{1-e^2}}{\mu}\left\{\frac{r_l'\sin\lambda_l'}{r_l'^3} + \frac{r\sin\lambda - r_l'\sin\lambda_l'}{\{r^2 - 2rr_l'\cos(\lambda-\lambda_l') + r_l'^2\}^{\frac{3}{2}}}\right\}$$

$$\frac{e\,\mathrm{d}\varpi}{\mathrm{d}v} = -\frac{r\sin\lambda}{\mu\sqrt{1-e^2}}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) + \frac{m_l a r\sqrt{1-e^2}}{\mu}\left\{\frac{r_l'\cos\lambda_l'}{r_l'^3} + \frac{r\cos\lambda - r_l'\cos\lambda_l'}{\{r^2 - 2rr_l'\cos(\lambda-\lambda_l') + r_l'^2\}^{\frac{3}{2}}}\right\}$$

$$\frac{\mathrm{d}\varepsilon - \mathrm{d}\varpi}{\mathrm{d}v} = -\frac{r^2}{a^2\sqrt{1-e^2}}\frac{\mathrm{d}\varpi}{\mathrm{d}v} - \left\{\frac{r}{a(1-e^2)} + 1\right\}\sin v\,\frac{\mathrm{d}e}{\mathrm{d}v}$$

$$\mathrm{d}a = -\frac{2a^2}{\mu}\,\mathrm{d}R$$

and since $r^2\,\mathrm{d}\lambda = \sqrt{\mu a(1-e^2)}\,\mathrm{d}t = a\sqrt{1-e^2}\,r\,\mathrm{d}v$

$$\frac{\mathrm{d}\imath}{\mathrm{d}v} + \frac{a\sqrt{1-e^2}\,r\cos\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}v} + \frac{a\sqrt{1-e^2}\,r\sin\lambda}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0.$$

The last six equations serve to determine the perturbations of a comet.

Let $(\Delta e)_n$ be the variation of any element e during the variation Δv of v at any given epoch n, neglecting the square and higher powers of Δv,

$$(\Delta e)_n = \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_n \Delta v$$

If the values of $(\Delta e)_n$ be calculated for the epochs $0, 1, 2 \ldots m$ corresponding to the values v, $v + i\,\Delta v$, $v + 2i\,\Delta v$, &c., differing from each other by $i\,\Delta v$, then the whole variation (δe) of v corresponding to the variation $i\,\Delta v$ of v

$$= i\left\{(\Delta e)_0 + (\Delta e)_1 \ldots\ldots + (\Delta e)_{m-1}\right\}$$

$$+ \frac{i-1}{2}\left\{(\Delta e)_m - (\Delta e)_0\right\} - \frac{(i-1)(i+1)}{12i}\left\{(\Delta^2 e)_m - (\Delta^2 e)_0\right\} + \text{&c.}$$

$$= i\left\{\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0 + \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_1 \ldots\ldots + \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_{m-1}\right\}\Delta v$$

$$+ \frac{i-1}{2}\left\{\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_m - \left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0\right\}\Delta v - \frac{(i-1)(i+1)}{12i}\left\{\Delta\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_m - \Delta\left(\frac{\mathrm{d}e}{\mathrm{d}v}\right)_0\right\}\Delta v.$$

When the interval $\Delta\, v$ is indefinitely diminished, $i\, \Delta\, v$ is still equal to the variation of v between the epochs for which the quantities $\left(\frac{de}{dv}\right)_0$, $\left(\frac{de}{dv}\right)_1$ &c. are calculated, and

$$\delta\, e = i\, \Delta\, v \left\{ \left(\frac{de}{dv}\right)_0 + \left(\frac{de}{dv}\right)_1 \ldots\ldots + \left(\frac{de}{dv}\right)_{m-1} + \frac{1}{2}\left\{\left(\frac{de}{dv}\right)_m - \left(\frac{de}{dv}\right)_0\right\} \right.$$
$$\left. - \frac{1}{12}\left\{\Delta\left(\frac{de}{dv}\right)_m - \Delta\left(\frac{de}{dv}\right)_0\right\} + \text{&c.} \right\}$$

If the radius be taken for unity, and $i\, \Delta\, v$ is the mth part of the circumference; $i\, \Delta\, v = \frac{2 \times 3{\cdot}14159}{m}$, or, in other words, the resulting values of $\delta\, e$ and $\delta\, a$ in the equations given above must be multiplied by 2×3.14159, and divided by 360° expressed in the same unit as $i\, \Delta\, v$.

n is equal to the angular circumference divided by the periodic time expressed in the same unit as t; so that if a degree be taken as the unity of angular circumference, $n = 360°$ divided by the periodic time expressed in the same unit as t.

In the elliptic movement or first approximation

$\lambda = n\, t + \varepsilon +$ a series of sines of arcs multiples of $n\, t$ &c.

$\lambda_1 = n_1\, t + \varepsilon_1 +$ a series of sines of arcs multiples of $n_1\, t$ &c.

$\frac{a}{r} =$ constant $+$ a series of cosines.

$s =$ a series of sines.

$s_1 =$ a series of sines.

These values being substituted in the equations of p. 334 give $\frac{de}{d\lambda}$, $\frac{dh}{d\lambda}$ and $\frac{di}{d\lambda}$ each equal to a series of sines without any constant quantity, and $\frac{d\varpi}{d\lambda}$, $\frac{dv}{d\lambda}$ and $\frac{d\varepsilon}{d\lambda}$ each equal to a series of cosines $+$ a constant quantity.

In the second approximation the values of λ, r and s retain the same form; and it is easy to see from the form of the expressions for $\frac{de}{d\lambda}$, $\frac{dh}{d\lambda}$, &c. p. 334, that the form of the values of these quantities is not altered however far the approximation be carried.

If the sun or primary be a spheroid, ω the angle which the plane of the sun's equator makes with the plane of the orbit of the planet; and if the longitude be reckoned from the line of intersection of the sun's equator with the orbit of the planet; R is increased by the quantity $c\left\{\frac{3\sin^2\omega\sin^2\lambda - 1}{r^3}\right\}$, c being a constant dependent upon the figure of the sun; but the partial differential coefficients of this quantity, which are introduced into the values of d e, d ϖ, &c. do not change the form of the expressions for those quantities.

If the planet move in a medium which resists according to any power n of the velocity, if c be a constant and v the velocity, the term $2c\int v^{n+1}\,\mathrm{d}\,t$ must be added to $2\int \mathrm{d}\,R$,

$$c\,v^{n-1}\left\{(1+s^2)\frac{\mathrm{d}\,r'}{\mathrm{d}\,t} + r's\frac{\mathrm{d}\,s}{\mathrm{d}\,t}\right\} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r'}\right),$$

$$c\,v^{n-1}\,r'^2\frac{\mathrm{d}\,\lambda'}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right), \quad \text{and } c\,v^{n-1}\,r'\frac{\mathrm{d}.r's}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right)$$

in the equations of p. 330.

If the orbit of the planet be supposed to coincide with the plane $x\,y$, so that $s = 0$, then by the equations of p. 337 after reductions

$$\mathrm{d}\,a = -2c\,a^2\left(\frac{\mu}{a}\right)^{\frac{n+1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}}\frac{(1-e\cos v)}{n}\,\mathrm{d}\,v$$

$$\mathrm{d}\,e = -2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\left(\frac{1-e^2}{n}\right)\cos v\,\mathrm{d}\,v$$

$$e\,\mathrm{d}\,\varpi = -2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\frac{\sqrt{1-e^2}}{n}\sin v\,\mathrm{d}\,v$$

$$\mathrm{d}\,\varepsilon - \mathrm{d}\,\varpi = 2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\frac{\sin v}{n}\left\{\frac{1-e^3\cos v}{e}\right\}\mathrm{d}\,v$$

The form of these equations differs from that which obtained before, now the variations of e, ϖ and ε are periodical, while that of a has a term which varies with the time. $\frac{\mathrm{d}\,e}{\mathrm{d}\,v}$ contains only odd powers of cos v and for that reason has no constant term. The periods of the periodic inequalities of all the elliptic constants due to the action of the resisting medium are fractional parts of the periodic time of the planet.

If the origin of t coincides with the instant of the perihelion passage, by LAGRANGE's theorem

$$\cos v = \cos nt - e\sin nt^2 - \frac{e^2}{2}\cdot\frac{\mathrm{d}.\sin nt^3}{\mathrm{d}.nt} - \frac{e^3}{2.3}\cdot\frac{\mathrm{d}^2.\sin nt^4}{(\mathrm{d}.nt)^2} - \frac{e^4}{2.3.4}\frac{\mathrm{d}^3\sin nt^5}{(\mathrm{d}.nt^3} - \&c.$$

$$\sin v = \sin nt + e\sin nt\cos nt + \frac{e^2}{2}\frac{\mathrm{d}.\sin nt^2\cos nt}{\mathrm{d}.nt} + \frac{e^3}{2.3}\frac{\mathrm{d}^2.\sin nt^3\cos nt}{(\mathrm{d}.nt)^2} + \frac{e^4}{2.3.4}\frac{\mathrm{d}^3.\sin nt^4\cos nt}{(\mathrm{d}.nt)^3}$$

$$\sin nt^2 = \frac{1-\cos 2nt}{2},\quad \sin nt^3 = \frac{3\sin nt - \sin 3nt}{4}$$

$$\sin nt^4 = \frac{3-4\cos 2nt+\cos 4nt}{8},\quad \sin nt^5 = \frac{10\sin nt - 5\sin 3nt + \sin 5nt}{16}$$

$$\frac{\mathrm{d}.\sin nt^3}{\mathrm{d}.nt} = \frac{3\cos nt - 3\cos 3nt}{4}$$

$$\frac{\mathrm{d}.\sin nt^4}{\mathrm{d}.nt} = \frac{2\sin 2nt - \sin 4nt}{2},\quad \frac{\mathrm{d}^2.\sin nt^4}{(\mathrm{d}.nt)^2} = 2\cos 2nt - 2\cos 4nt$$

$$\frac{\mathrm{d}.\sin nt^5}{\mathrm{d}.nt} = \frac{10\cos nt - 15\cos 3nt + 5\cos 5nt}{16}$$

$$\frac{\mathrm{d}^2\sin nt^5}{(\mathrm{d}.nt)^2} = \frac{-10\sin nt + 45\sin 3nt - 25\sin 5nt}{16}$$

$$\frac{\mathrm{d}^3.\sin nt^5}{(\mathrm{d}.nt)} = \frac{-10\cos nt + 135\cos 3nt - 125\cos 5nt}{16}$$

$$\cos v = \cos nt - \frac{e}{2} + \frac{e}{2}\cos 2nt - \frac{e^2}{2}\left\{\frac{3\cos nt - 3\cos 3nt}{4}\right\} - \frac{e^3}{2.3}\left\{2\cos 2nt - 2\cos 4nt\right\}$$
$$- \frac{e^4}{2.3.4}\left\{\frac{-10\cos nt + 135\cos 3nt - 125\cos 5nt}{16}\right\}$$

If the origin of the time does not coincide with the perihelion passage, $nt + \varepsilon - \varpi$ must be substituted for nt, but as ε always accompanies nt, it may be suppressed at present for convenience, and afterwards replaced.

$$\cos v = \left\{1 - \frac{3}{8}e^2 + \frac{5e^4}{192}\right\}\cos(nt-\varpi) - \frac{e}{2} + \frac{e}{2}\left\{1 - \frac{2e^2}{3}\right\}\cos(2nt - 2\varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{15}{16}e^2\right\}\cos(3nt - 3\varpi) + \frac{e^3}{3}\cos(4nt - 4\varpi) + \frac{125}{384}e^4\cos(5nt - 5\varpi)$$

$$\sin nt^2\cos nt = \frac{\mathrm{d}.\sin nt^3}{3\,\mathrm{d}.nt} = \frac{\cos nt - \cos 3nt}{4},\quad \frac{\mathrm{d}.\sin nt^2\cos nt}{\mathrm{d}.nt} = \frac{-\sin nt + 3\sin 3nt}{4}$$

$$\sin n t^3 \cos n t = \frac{\mathrm{d}.\sin n t^4}{4\,\mathrm{d}.nt} = \frac{2\sin 2nt - \sin 4nt}{8}, \qquad \frac{\mathrm{d}.\sin nt^3\cos nt}{\mathrm{d}.nt} = \frac{\cos 2nt - \cos 4nt}{2}$$

$$\frac{\mathrm{d}^2.\sin nt^3\cos nt}{(\mathrm{d}.nt)^2} = -\sin 2nt + 2\sin 4nt$$

$$\sin n t^4 \cos nt = \frac{\mathrm{d}.\sin nt^5}{5\,\mathrm{d}.nt} = \frac{2\cos nt - 3\cos 3nt + \cos 5nt}{16}$$

$$\frac{\mathrm{d}.\sin nt^4\cos nt}{\mathrm{d}.nt} = \frac{-2\sin nt + 9\sin 3nt - 5\sin 5nt}{16}$$

$$\frac{\mathrm{d}^2.\sin nt^4\cos nt}{(\mathrm{d}.nt)^2} = \frac{-2\cos nt + 27\cos 3nt - 25\cos 5nt}{16}$$

$$\frac{\mathrm{d}^3.\sin nt^4\cos nt}{(\mathrm{d}.nt)^3} = \frac{2\sin nt - 81\sin 3nt + 125\sin 5nt}{16}$$

$$\sin v = \sin nt + \frac{e}{2}\sin 2nt + \frac{e^2}{2}\left\{\frac{-\sin nt + 3\sin 3nt}{4}\right\} + \frac{e^3}{2.3}\{-\sin 2nt + 2\sin 4nt\}$$
$$+ \frac{e^4}{2.3.4}\left\{\frac{2\sin nt - 81\sin 3nt + 125\sin 5nt}{16}\right\}$$

and replacing nt by $nt - \varpi$

$$\sin v = \left\{1 - \frac{3e^2}{8} + \frac{5}{192}e^4\right\}\sin(nt-\varpi) + \frac{e}{2}\left\{1 - 2\frac{e^2}{3}\right\}\sin(2nt - 2\varpi)$$
$$+ \left\{\frac{2e^2}{8} - \frac{4}{192}e^4\right\}\sin(nt-\varpi) + \frac{e^3}{6}\sin(2nt - 2\varpi)$$
$$+ \frac{3e^2}{8}\left\{1 - \frac{15}{16}e^2\right\}\sin(3nt - 3\varpi) + \frac{e^3}{3}\sin(4nt - 4\varpi) + \frac{125}{384}e^4\sin(5nt - 5\varpi)$$
$$+ \frac{3}{8}\cdot\frac{6}{16}e^4\sin(3nt - 3\varpi)$$

$$\cos(v+\varpi) = \left\{1 - \frac{e^2}{4} + \frac{e^4}{64}\right\}\cos nt - \frac{e}{2}\cos\varpi + \frac{e}{2}\left\{1 - \frac{e^2}{2}\right\}\cos(2nt - \varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{3}{4}e^2\right\}\cos(3nt - 2\varpi) + \frac{e^3}{3}\cos(4nt - 3\varpi) + \frac{125}{384}e^4\cos(5nt - 4\varpi)$$
$$- \frac{e^3}{8}\left\{1 - \frac{e^2}{12}\right\}\cos(nt - 2\varpi) - \frac{e^3}{12}\cos(2nt - 3\varpi) - \frac{9}{128}e^4\cos(3nt - 4\varpi)$$

$$\sin(v+\varpi) = \left\{1 - \frac{e^2}{4} + \frac{e^4}{64}\right\}\sin nt - \frac{e}{2}\sin\varpi + \frac{e}{2}\left\{1 - \frac{e^2}{2}\right\}\sin(2nt - \varpi)$$
$$+ \frac{3}{8}e^2\left\{1 - \frac{3}{4}e^2\right\}\sin(3nt - 2\varpi) + \frac{e^3}{3}\sin(4nt - 3\varpi) + \frac{125}{384}e^4\sin(5nt - 4\varpi)$$

$$+\frac{e^2}{8}\left(1-\frac{e^2}{12}\right)\sin(nt-2\varpi)+\frac{e^3}{12}\sin(2nt-3\varpi)+\frac{9}{128}e^4\sin(3nt-4\varpi)$$

$$\cos(\upsilon-\varpi)=\left(1-\frac{e^2}{4}\right)\cos(nt-2\varpi)-\frac{e}{2}\cos\varpi+\frac{e}{2}\cos(2nt-3\varpi)+\frac{3}{8}e^2\cos(3nt-4\varpi)-\frac{e^2}{8}\cos nt$$

$$\sin(\upsilon-\varpi)=\left(1-\frac{e^2}{4}\right)\sin(nt-2\varpi)+\frac{e}{2}\sin\varpi+\frac{e}{2}\sin(2nt-3\varpi)+\frac{3}{8}e^2\sin(3nt-4\varpi)+\frac{e^2}{8}\sin nt$$

$$r\cos\lambda=r\cos(\lambda-\varpi)\cos\varpi-r\sin(\lambda-\varpi)\sin\varpi=a\{(\cos\upsilon-e)\cos\varpi-(1-e^2)^{\frac{1}{2}}\sin\upsilon\sin\varpi\}$$

$$r\sin\lambda=r\sin(\lambda-\varpi)\cos\varpi+r\cos(\lambda-\varpi)\sin\varpi=a\{(1-e^2)^{\frac{1}{2}}\sin\upsilon\cos\varpi+(\cos\upsilon-e)\sin\varpi\}$$

$$r\cos\lambda=a\left\{\left(1-\frac{e^2}{4}-\frac{e^4}{16}\right)\cos(\upsilon+\varpi)-e\cos\varpi+\frac{e^2}{4}\left(1+\frac{e^2}{4}\right)\cos(\upsilon-\varpi)\right\}$$

$$r\sin\lambda=a\left\{\left(1-\frac{e^2}{4}-\frac{e^4}{16}\right)\sin(\upsilon+\varpi)-e\sin\varpi-\frac{e^2}{4}\left(1+\frac{e^2}{4}\right)\sin(\upsilon-\varpi)\right\}$$

$$\begin{matrix}r\cos\lambda\\ r\sin\lambda\end{matrix}=a\left\{\left(1-\frac{e^2}{2}-\frac{e^4}{64}\right)\begin{matrix}\cos\\ \sin\end{matrix}nt-\frac{3e}{2}\begin{matrix}\cos\\ \sin\end{matrix}\varpi+\frac{e}{2}\left(1-\frac{3}{4}e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi)\right.$$

$$+\frac{3}{8}e^2\left(1-e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2\varpi)+\frac{e^3}{3}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-3\varpi)+\frac{125}{384}e^4\begin{matrix}\cos\\ \sin\end{matrix}(5nt-4\varpi)$$

$$\left.\begin{matrix}+\\ -\end{matrix}\frac{e^2}{8}\left(1+\frac{e^2}{3}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-2\varpi)\begin{matrix}+\\ -\end{matrix}\frac{e^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3\varpi)\begin{matrix}+\\ -\end{matrix}\frac{3}{128}e^4\begin{matrix}\cos\\ \sin\end{matrix}(3nt-4\varpi)\,\&c.\right\}$$

$$rr_{,}\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda-\lambda_{,}\}=aa_{,}\left\{\left(1-\frac{e^2}{2}-\frac{e^4}{64}\right)\left(1-\frac{e_{,}^2}{2}-\frac{e_{,}^4}{64}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_{,}t)\right.$$

$$\begin{matrix}-\\ +\end{matrix}\frac{3}{2}e\left(1-\frac{e_{,}^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(n_{,}t-\varpi)+\frac{e}{2}\left(1-\frac{3}{4}e^2\right)\left(1-\frac{e_{,}^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-n_{,}t-\varpi)$$

$$+\frac{3}{8}e^2\left(1-e^2\right)\left(1-\frac{e_{,}^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(3nt-n_{,}t-2\varpi)+\frac{e^3}{3}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-n_{,}t-3\varpi)$$

$$+\frac{125}{384}e^4\begin{matrix}\cos\\ \sin\end{matrix}(5nt-n_{,}t-4\varpi)\begin{matrix}+\\ -\end{matrix}\frac{e^2}{8}\left(1+\frac{e^2}{3}\right)\left(1-\frac{e_{,}^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_{,}t-2\varpi)$$

$$\begin{matrix}+\\ -\end{matrix}\frac{e^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_{,}t-3\varpi)\begin{matrix}+\\ -\end{matrix}\frac{3}{128}e^4\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_{,}t-4\varpi)$$

$$-\frac{3}{2}e_{,}\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-\varpi_{,})+\frac{9}{4}ee_{,}\begin{matrix}\cos\\ \sin\end{matrix}(\varpi-\varpi_{,})$$

$$-\frac{3}{4}ee_{,}\left(1-\frac{3}{4}e^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi-\varpi_{,})-\frac{9}{16}e^2e_{,}\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2\varpi-\varpi_{,})$$

$$\left.-\frac{e^3e_{,}}{2}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-3\varpi-\varpi_{,})\begin{matrix}-\\ +\end{matrix}\frac{3}{16}e^2e_{,}\begin{matrix}\cos\\ \sin\end{matrix}(nt-2\varpi+\varpi_{,})\begin{matrix}-\\ +\end{matrix}\frac{e^3e_{,}}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3\varpi+\varpi_{,})\right.$$

$$+\frac{e_1}{2}\left(1-\frac{3}{4}e_1^2\right)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-2n_1t+\varpi_1)\begin{matrix}-\\+\end{matrix}\frac{3}{4}ee_1\left(1-\frac{3}{4}e_1^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2n_1t-\varpi-\varpi_1)$$

$$+\frac{ee_1}{4}\left(1-\frac{3}{4}e^2\right)\left(1-\frac{3}{4}e_1^2\right)\begin{matrix}\cos\\ \sin\end{matrix}(2nt-2n_1t-\varpi+\varpi_1)+\frac{3}{16}e^2e_1\begin{matrix}\cos\\ \sin\end{matrix}(3nt-2n_1t-2\varpi+\varpi_1)$$

$$+\frac{e^3e_1}{6}\begin{matrix}\cos\\ \sin\end{matrix}(4nt-2n_1t-3\varpi+\varpi_1)\begin{matrix}+\\-\end{matrix}\frac{e^2e_1}{16}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_1t-2\varpi-\varpi_1)$$

$$\begin{matrix}+\\-\end{matrix}\frac{e^3e_1}{48}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_1t-3\varpi-\varpi_1)+\frac{3}{8}e_1^2(1-e_1^2)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt-3n_1t+2\varpi_1)$$

$$\begin{matrix}-\\+\end{matrix}\frac{9}{16}ee_1^2\begin{matrix}\cos\\ \sin\end{matrix}(3n_1t-\varpi-2\varpi_1)+\frac{3}{16}ee_1^2\begin{matrix}\cos\\ \sin\end{matrix}(2nt-3n_1t-\varpi+2\varpi_1)$$

$$+\frac{9}{64}e^2e_1^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt-3n_1t-2\varpi+2\varpi_1)\begin{matrix}+\\-\end{matrix}\frac{3}{64}e^2e_1^2\begin{matrix}\cos\\ \sin\end{matrix}(nt+3n_1t-2\varpi-2\varpi_1)$$

$$+\frac{e^3}{3}\begin{matrix}\cos\\ \sin\end{matrix}(nt-4n_1t+3\varpi_1)\begin{matrix}-\\+\end{matrix}\frac{ee_1^3}{2}\begin{matrix}\cos\\ \sin\end{matrix}(4n_1t-\varpi-3\varpi_1)$$

$$+\frac{ee_1^3}{6}\begin{matrix}\cos\\ \sin\end{matrix}(2nt-4n_1t-\varpi+3\varpi_1)+\frac{125}{384}e_1^4\begin{matrix}\cos\\ \sin\end{matrix}(nt-5n_1t+4\varpi_1)$$

$$+\frac{e_1^2}{8}\left(1+\frac{e^2}{3}e_1^2\right)\left(1-\frac{e^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_1t-2\varpi_1)-\frac{3}{16}ee_1^2\begin{matrix}\cos\\ \sin\end{matrix}(n_1t+\varpi-2\varpi_1)$$

$$+\frac{ee_1^2}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_1t-\varpi-2\varpi_1)+\frac{3}{64}e^2e_1^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_1t-2\varpi-2\varpi_1)$$

$$\begin{matrix}+\\-\end{matrix}\frac{e^2e_1^2}{64}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_1t-2\varpi+2\varpi_1)+\frac{e_1^3}{24}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_1t-3\varpi_1)$$

$$\begin{matrix}-\\+\end{matrix}\frac{ee_1^3}{16}\begin{matrix}\cos\\ \sin\end{matrix}(2n_1t+\varpi-3\varpi_1)+\frac{ee_1^3}{48}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_1t-\varpi-3\varpi_1)$$

$$+\frac{}{128}e_1^4\begin{matrix}\cos\\ \sin\end{matrix}(nt+3n_1t-4\varpi_1)+\&c.$$

$$rr_1\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda+\lambda_1\}=aa_1\left\{\left(1-\frac{e^2+e_1^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_1t)\right.$$

$$-\frac{3}{2}e\begin{matrix}\cos\\ \sin\end{matrix}(n_1t+\varpi)+\frac{e}{2}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_1t-\varpi)$$

$$+\frac{3}{8}e^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_1t-2\varpi)\begin{matrix}+\\-\end{matrix}\frac{e^2}{8}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_1t-2\varpi)$$

$$-\frac{3}{2}e_1\begin{matrix}\cos\\ \sin\end{matrix}(nt+\varpi_1)+\frac{9}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(\varpi+\varpi_1)-\frac{3}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi+\varpi_1)$$

$$+\frac{e_1}{2}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_1t-\varpi_1)-\frac{3}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(2n_1t+\varpi-\varpi_1)+\frac{ee_1}{4}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2nt-\varpi-\varpi_1)+\&c.$$

$$rr_1\begin{matrix}\cos\\ \sin\end{matrix}\{\lambda+\lambda_1-2\nu\}=aa_1\left\{\left(1-\frac{e^2+e_1^2}{2}\right)\begin{matrix}\cos\\ \sin\end{matrix}(nt+n_1t-2\nu)\right.$$

$$-\frac{3}{2}e\begin{matrix}\cos\\ \sin\end{matrix}(n_1t+\varpi-2\nu)+\frac{e}{2}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+n_1t-\varpi-2\nu)$$

$$+\frac{3}{8}e^2\begin{matrix}\cos\\ \sin\end{matrix}(3nt+n_1t-2\varpi-2\nu)\pm\frac{e^2}{8}\begin{matrix}\cos\\ \sin\end{matrix}(nt-n_1t-2\varpi+2\nu)$$

$$-\frac{3}{2}e_1\begin{matrix}\cos\\ \sin\end{matrix}(nt+\varpi_1-2\nu)+\frac{9}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(\varpi+\varpi_1-2\nu)$$

$$-\frac{3}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(2nt-\varpi+\varpi_1-2\nu)+\frac{e_1}{2}\begin{matrix}\cos\\ \sin\end{matrix}(nt+2n_1t-\varpi_1-2\nu)$$

$$-\frac{3}{4}ee_1\begin{matrix}\cos\\ \sin\end{matrix}(2n_1t+\varpi-\varpi_1-2\nu)+\frac{ee_1}{4}\begin{matrix}\cos\\ \sin\end{matrix}(2nt+2n_1t-\varpi-\varpi_1-2\nu)+\&c.$$

$$r=a\left\{1+\frac{e^2}{2}-e\left(1-\frac{3}{8}e^2\right)\cos(nt-\varpi)-\frac{e^2}{2}\left(1-\frac{2e^2}{3}\right)\cos(2nt-2\varpi)\right.$$

$$-\frac{3}{8}e^3\cos(3nt-3\varpi)-\frac{e^4}{3}\cos(4nt-4\varpi)+\&c.$$

$$r^2=a^2\left\{1+\frac{3e^2}{2}-2e\left(1-\frac{e^2}{8}\right)\cos(nt-\varpi)-\frac{e^2}{2}\left(1-\frac{e^2}{3}\right)\cos(2nt-2\varpi)\right.$$

$$-\frac{e^3}{4}\cos(3nt-3\varpi)-\frac{e^4}{6}\cos(4nt-4\varpi)+\&c.$$

$$r^{-3}=a^{-3}\left\{1+\frac{3}{2}e^2\left(1+\frac{e^2}{2}\right)+3e\left(1+\frac{9}{8}e^2\right)\cos(nt-\varpi)\right.$$

$$+\frac{9}{2}e^2\left(1-\frac{e^2}{18}\right)\cos 2nt-2\varpi+\frac{53}{8}e^3\cos(3nt-3\varpi)$$

$$+\frac{31}{4}e^4\cos(4nt-4\varpi)+\&c.$$

If the coefficient of $\cos n\theta$ in the development of $\left\{1-\frac{2a}{a_1}\cos\theta+\frac{a^2}{a_1^2}\right\}^{-m}$ according to cosines of arcs mulitples of θ is called $b_{2m,n}$

$$b_{1,0}=1+\left(\frac{1}{2}\right)^2\frac{a^2}{a_1^2}+\left(\frac{1.3}{2.4}\right)^2\frac{a^4}{a_1^4}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{a^6}{a_1^6}+\&c.$$

$$b_{1,1} = \frac{a}{a_1} + \frac{1.3}{2.4}\frac{a^3}{a_1^3} + \frac{1.3.3.5}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{1,2} = \frac{3}{4}\frac{a^2}{a_1^2} + \frac{1.3.5}{2.4.6}\frac{a^4}{a_1^4} + \frac{1.3.3.5.7}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{1,3} = \frac{3.5}{4.6}\frac{a^3}{a_1^3} + \frac{1.3.5.7}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{1.3.3.5.7.9}{2.4.4.6.8.10}\frac{a^7}{a_1^7} + \&c.$$

$$b_{3,0} = 1 + \left(\frac{3}{2}\right)^2\frac{a^2}{a_1^2} + \left(\frac{3.5}{2.4}\right)^2\frac{a^4}{a_1^4} + \left(\frac{3.5.7}{2.4.6}\right)^2\frac{a^6}{a_1^6} + \&c.$$

$$b_{3,1} = \frac{3a}{a_1} + \frac{3.3.5}{2.4}\frac{a^3}{a_1^3} + \frac{3.5.3.5.7}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{3,2} = \frac{3.5}{4}\frac{a^2}{a_1^2} + \frac{3.3.5.7}{2.4.6}\frac{a^4}{a_1^4} + \frac{3.5.3.5.7}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{3,3} = \frac{3.5.7}{4.6}\frac{a^3}{a_1^3} + \frac{3.3.5.7.9}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{3.5.3.5.7.9.11}{2.4.4.6.8.10}\frac{a^7}{a_1^7} + \&c.$$

$$b_{5,0} = 1 + \left(\frac{5}{2}\right)^2\frac{a^2}{a_1^2} + \left(\frac{5.7}{2.4}\right)^2\frac{a^4}{a_1^4} + \left(\frac{5.7.9}{2.4.6}\right)^2\frac{a^6}{a_1^6} + \&c.$$

$$b_{5,1} = 5\frac{a}{a_1} + \frac{5.5.7}{2.4}\frac{a^3}{a_1^3} + \frac{5.7.5.7.9}{2.4.4.6}\frac{a^5}{a_1^5} + \&c.$$

$$b_{5,2} = \frac{5.7}{4}\frac{a^2}{a_1^2} + \frac{5.5.7.9}{2.4.6}\frac{a^4}{a_1^4} + \frac{5.7.5.7.9.11}{2.4.4.6.8}\frac{a^6}{a_1^6} + \&c.$$

$$b_{5,3} = \frac{5.7.9}{4.6}\frac{a^3}{a_1^3} + \frac{5.5.7.9.11}{2.4.6.8}\frac{a^5}{a_1^5} + \frac{5.7.5.7.9.11.13}{2.4.4.6.8.10}\frac{a^7}{a_1^7}$$

$$b_{3,1} = 3\frac{a}{a_1}\left\{b_{5,0} - \frac{1}{2}b_{5,2}\right\} \qquad b_{1,1} = \frac{a}{a_1}\left\{b_{3,0} - \frac{1}{2}b_{3,2}\right\}$$

$$2\,b_{3,2} = \frac{3}{2}\frac{a}{a_1}\left\{b_{5,1} - b_{5,3}\right\} \qquad 2\,b_{1,2} = \frac{1}{2}\frac{a}{a_1}\left\{b_{3,1} - b_{3,3}\right\}$$

$$3\,b_{3,3} = \frac{3}{2}\frac{a}{a_1}\left\{b_{5,2} - b_{5,4}\right\} \qquad 3\,b_{1,3} = \frac{1}{2}\frac{a}{a_1}\left\{b_{3,2} - b_{3,4}\right\}$$

$$R = m'\left\{\frac{r r_1' \cos(\lambda - \lambda_1')}{r_1'^3} - \frac{1}{\{r^2 - 2 r r_1' \cos(\lambda - \lambda_1') + r_1'^2\}^{\frac{1}{2}}}\right\}$$

and since

$$r r_1' \cos(\lambda - \lambda_1') = r r_1'\left\{\cos^2\tfrac{i}{2}\cos(\lambda - \lambda_1) + \sin^2\tfrac{i}{2}\cos(\lambda + \lambda_1 - 2\nu)\right\}, \text{ p. 330;}$$

$$R = m_{,}\left\{\frac{r r_{,}\left\{\cos^2\frac{\iota_{,}}{2}\cos(\lambda-\lambda_{,}) + \sin^2\frac{\iota_{,}}{2}\cos(\lambda+\lambda_{,}-2\nu_{,})\right\}}{r_{,}^3}\right.$$

$$\left. - \frac{1}{\left\{r^2 - 2 r r_{,}\left\{\cos^2\frac{\iota_{,}}{2}\cos(\lambda-\lambda_{,}) + \sin^2\frac{\iota_{,}}{2}\cos(\lambda+\lambda_{,}-2\nu_{,})\right\} + r_{,}^2\right\}^{\frac{1}{2}}}\right\}$$

Neglecting the cubes and higher powers of e

$$R = m_{,}\frac{a}{a_{,}^2}\left\{\left(1-\sin^2\frac{\iota_{,}}{2}-\frac{e^2+e_{,}^2}{2}\right)\cos(nt-n_{,}t) - \frac{3}{2}e\cos(n_{,}t-\varpi) - \frac{3}{2}e_{,}\cos(nt-\varpi_{,})\right.$$

$$+\frac{e}{2}\cos(2nt-n_{,}t-\varpi) + \frac{e_{,}}{2}\cos(nt-2n_{,}t+\varpi_{,}) + \frac{3}{8}e^2\cos(3nt-n_{,}t-2\varpi)$$

$$+\frac{3}{8}e_{,}^2\cos(nt-3n_{,}t+2\varpi_{,}) + \frac{e^2}{8}\cos(nt+n_{,}t-2\varpi) + \frac{e_{,}^2}{8}\cos(nt+n_{,}t-2\varpi_{,})$$

$$+\frac{9}{4}ee_{,}\cos(\varpi-\varpi_{,}) - \frac{3}{4}ee_{,}\cos(2nt-\varpi-\varpi_{,}) - \frac{3}{4}ee_{,}\cos(2n_{,}t-\varpi-\varpi_{,})$$

$$\left.+\frac{e e_{,}}{4}\cos(2nt-2n_{,}t-\varpi+\varpi_{,}) + \sin^2\frac{\iota}{2}\cos(nt+n_{,}t-2\nu)\right\}\left\{1+\frac{3}{2}e_{,}^2\right.$$

$$\left.+3e_{,}\cos(n_{,}t-\varpi_{,}) + \frac{9}{2}e_{,}^2\cos(2n_{,}t-2\varpi_{,}) + \&c.\right\}$$

$$-m_{,}\left\{a^2\left\{1+\frac{3}{2}e^2-2e\cos(nt-\varpi)-\frac{e^2}{2}\cos(2nt-2\varpi)\right\}\right.$$

$$-2aa_{,}\left\{\left(1-\sin^2\frac{\iota_{,}}{2}-\frac{e^2+e_{,}^2}{2}\right)\cos(nt-n_{,}t) - \frac{3}{2}e\cos(n_{,}t-\varpi)\right.$$

$$-\frac{3}{2}e_{,}\cos(nt-\varpi_{,}) + \frac{e}{2}\cos(2nt-n_{,}t-\varpi) + \frac{e_{,}}{2}\cos(nt-2n_{,}t+\varpi_{,})$$

$$+\frac{3}{8}e^2\cos(3nt-n_{,}t-2\varpi) + \frac{3}{8}e_{,}^2\cos(nt-3n_{,}t+2\varpi_{,}) + \frac{e^2}{8}\cos(nt+n_{,}t-2\varpi)$$

$$+\frac{e_{,}^2}{8}\cos(nt+n_{,}t-2\varpi_{,}) + \frac{9}{4}ee_{,}\cos(\varpi-\varpi_{,}) - \frac{3}{4}ee_{,}\cos(2nt-\varpi-\varpi_{,})$$

$$-\frac{3}{4}ee_{,}\cos(2n_{,}t-\varpi-\varpi_{,}) + \frac{e e_{,}}{4}\cos(2nt-2n_{,}t-\varpi+\varpi_{,})$$

$$\left.+\sin^2\frac{\iota}{2}\cos(nt+n_{,}t-2\nu)\right\} + a_{,}^2\left\{1+\frac{3}{2}e_{,}^2\right.$$

$$\left.\left.-2e_{,}\cos(n_{,}t-\varpi_{,}) - \frac{e_{,}^2}{2}\cos(2n_{,}t-2\varpi_{,}) + \&c.\right\}\right\}^{\frac{1}{2}}$$

$= \text{terms independent of } b$

$$- \frac{m_{,}}{a_{,}}\left\{ b_{1,0} + b_{1,1}\cos(nt - n_{,}t) + b_{1,2}\cos(2nt - 2n_{,}t) + \&c. \right\}$$

$$+ \frac{m_{,}}{2\,a_{,}^{3}}\left\{ b_{3,0} + b_{3,1}\cos(nt - n_{,}t) + b_{3,2}\cos(2nt - 2n_{,}t)\ \&c. \right\}\left\{ a^2 \left\{ \frac{3}{2}e^2 \right.\right.$$

$$\left. - 2e\cos(nt - \varpi) - \frac{e^2}{2}\cos(2nt - 2\varpi) \right\} - 2aa_{,}\left\{ \left(-\sin^2\frac{i_{,}}{2} - \frac{e^2 + e_{,}^2}{2} \right)\cos(nt - n_{,}t) \right.$$

$$- \frac{3}{2}e\cos(n_{,}t - \varpi) - \frac{3}{2}e_{,}\cos(nt - \varpi_{,}) + \frac{e}{2}\cos(2nt - n_{,}t - \varpi)$$

$$+ \frac{e_{,}}{2}\cos(nt - 2n_{,}t + \varpi_{,}) + \frac{3}{8}e^2\cos(3nt - n_{,}t - 2\varpi) + \frac{3}{8}e_{,}^2\cos(nt - 3n_{,}t + 2\varpi_{,})$$

$$+ \frac{e^2}{8}\cos(nt + n_{,}t - 2\varpi) + \frac{e_{,}^2}{8}\cos(nt + n_{,}t - 2\varpi_{,}) + \frac{9}{4}ee_{,}\cos(\varpi - \varpi_{,})$$

$$- \frac{3}{4}ee_{,}\cos(2nt - \varpi - \varpi_{,}) - \frac{3}{4}ee_{,}\cos(2n_{,}t - \varpi - \varpi_{,}) + \frac{ee_{,}}{4}\cos(2nt - 2n_{,}t - \varpi + \varpi_{,})$$

$$\left. + \sin^2\frac{i_{,}}{2}\cos(nt + n_{,}t - 2\nu_{,}) \right\} + a_{,}^2\left\{ \frac{3}{2}e_{,}^2 - 2e_{,}\cos(n_{,}t - \varpi_{,}) - \frac{e_{,}^2}{2}\cos(2n_{,}t - 2\varpi_{,})\&c. \right\}\left. \right\}$$

$$- \frac{1\cdot 3\,m_{,}}{2\cdot 4\,a_{,}^{5}}\left\{ b_{5,0} + b_{5,1}\cos(nt - n_{,}t) + b_{5,2}\cos(2nt - 2n_{,}t) + \&c. \right\}\left\{ 2a^2 e\cos(nt - \varpi) \right.$$

$$- 3aa_{,}e\cos(n_{,}t - \varpi) - 3aa_{,}e_{,}\cos(nt - \varpi_{,}) + aa_{,}e\cos(2nt - n_{,}t - \varpi)$$

$$\left. + a\,a_{,}\,e_{,}\cos(nt - 2n_{,}t + \varpi_{,}) + 2\,a_{,}^2\,e_{,}\cos(n_{,}t - \varpi_{,}) \right\}^2 + \&c.$$

$$\left\{ 2\,a^2\,e\cos(nt - \varpi) - 3\,a\,a_{,}\,e\cos(n_{,}t - \varpi) - 3\,a\,a_{,}\,e_{,}\cos(nt - \varpi_{,}) \right.$$

$$\left. + aa_{,}e\cos(2nt - n_{,}t - \varpi) + aa_{,}e_{,}\cos(nt - 2n_{,}t + \varpi_{,}) + 2\,a_{,}^2\,e_{,}\cos(n_{,}t - \varpi_{,}) \right\}^2$$

$$= 2\,a^4\,e^2 + 5\,a^2\,a_{,}^2\,(e^2 + e_{,}^2) + 2\,a_{,}^4\,e_{,}^2 + a^2(2\,a^2 - 3\,a_{,}^2)e^2\cos(2nt - 2\varpi)$$

$$+ \frac{9}{2}a^2a_{,}^2e^2\cos(2n_{,}t - 2\varpi) + \frac{9}{2}a^2a_{,}^2e_{,}^2\cos(2nt - 2\varpi_{,}) + \frac{a^2a_{,}^2}{2}e^2\cos(4nt - 2n_{,}t - 2\varpi)$$

$$+ \frac{a^2a_{,}^2}{2}e_{,}^2\cos(2nt - 4n_{,}t + 2\varpi_{,}) + a_{,}^2(2\,a_{,}^2 - 3\,a^2)e_{,}^2\cos(2n_{,}t - 2\varpi_{,})$$

$$- 4\,(a^2\,e^2 + a_{,}^2\,e_{,}^2)aa_{,}\cos(nt - n_{,}t) - 6\,a^3\,a_{,}\,e^2\cos(nt + n_{,}t - 2\varpi)$$

$$- 6\,aa_{,}^{3}\,e_{,}^{2}\cos(nt+n_{,}t-2\varpi_{,}) - 6\,(a^2+a_{,}^{2})\,aa_{,}ee_{,}\cos(\varpi-\varpi_{,})$$

$$-2(3a^2-a_{,}^{2})\,aa_{,}ee_{,}\cos(2nt-\varpi-\varpi_{,}) - 2(3a_{,}^{2}-a^2)\,aa_{,}ee_{,}\cos(2n_{,}t-\varpi-\varpi_{,})$$

$$+ 2\,a^3\,a_{,}e^2\cos(3nt-n_{,}t-2\varpi) + 2\,a\,a_{,}^{3}\,e_{,}^{2}\cos(nt-3n_{,}t+2\varpi_{,})$$

$$+ 2(a^2+a_{,}^{2})\,aa_{,}ee_{,}\cos(2nt-2n_{,}t-\varpi+\varpi_{,}) + 9\,a^2a_{,}^{2}ee_{,}\cos(nt-n_{,}t+\varpi-\varpi_{,})$$

$$+ 14\,a^2a_{,}^{2}\,e\,e_{,}\cos(nt+n_{,}t-\varpi-\varpi_{,}) - 3\,a^2a_{,}^{2}(e^2+e_{,}^{2})\cos(2nt-2n_{,}t)$$

$$- 3\,a^2a_{,}^{2}ee_{,}\cos(nt-3n_{,}t+\varpi+\varpi_{,}) - 3\,a^2a_{,}^{2}ee_{,}\cos(3nt-n_{,}t-\varpi-\varpi_{,})$$

$$- 2\,a^2a_{,}^{2}ee_{,}\cos(nt-n_{,}t-\varpi+\varpi_{,}) + a^2a_{,}^{2}ee_{,}\cos(3nt-3n_{,}t-\varpi+\varpi_{,})$$

Replacing ε which accompanies nt,

$$R = m_{,}\Big\{ -\frac{b_{1,0}}{a_{,}} + 3\,\frac{(a^2e^2+a_{,}^{2}e_{,}^{2})}{2\cdot2\,a_{,}^{3}}b_{3,0} + \frac{a\,a_{,}}{2\,a_{,}^{3}}\Big(\sin^2\frac{\imath_{,}}{2}+\frac{e^2+e_{,}^{2}}{2}\Big)b_{3,1}$$

$$-\frac{1\cdot3}{2\cdot4\cdot a_{,}^{5}}\Big(2a^4e^2+5a^2a_{,}^{2}(e^2+e_{,}^{2})+2a_{,}^{4}e_{,}^{2}\Big)b_{5,0}+\frac{1\cdot3\cdot2}{2\cdot4\,a_{,}^{5}}(a^2e^2+a_{,}^{2}e_{,}^{2})\,aa_{,}b_{5,1}$$

$$+\frac{1\cdot3\cdot3}{2\cdot4\cdot2}\frac{a^2a_{,}^{2}}{a_{,}^{5}}(e^2+e_{,}^{2})b_{5,2} \qquad [0^*]$$

$$+ m_{,}\Big\{\frac{a}{a_{,}^{2}}\Big\{\cos^2\frac{\imath_{,}}{2}-\frac{e^2+e_{,}^{2}}{2}\Big\} - \frac{b_{1,1}}{a_{,}} + \frac{a\,a_{,}}{a_{,}^{3}}\Big\{\sin^2\frac{\imath_{,}}{2}+\frac{e^2+e_{,}^{2}}{2}\Big\}\Big\{b_{3,0}+\tfrac{1}{2}\,b_{3,2}\Big\}$$

$$+\frac{3\,(a^2e^2+a_{,}^{2}e_{,}^{2})}{2\cdot2\,a_{,}^{3}}b_{3,1}+\frac{1\cdot3\cdot4}{2\cdot4\,a_{,}^{5}}(a^2e^2+a_{,}^{2}e_{,}^{2})\,aa_{,}b_{5,0}$$

$$-\frac{1\cdot3}{2\cdot4\,a_{,}^{5}}\Big\{2a^4e^2+\frac{7}{2}a^2a_{,}^{2}(e^2+e_{,})+2a_{,}^{4}e_{,}^{2}\Big\}b_{5,1}+\frac{1\cdot3\cdot2}{2\cdot4\,a_{,}^{5}}(a^2e^2+a_{,}^{2}e_{,}^{2})aa_{,}b_{5,2}$$

$$+\frac{1\cdot3\cdot3}{2\cdot4\cdot2}\frac{a^2a_{,}^{2}}{a_{,}^{5}}(e^2+e_{,}^{2})b_{5,3}\Big\}\cos(nt-n_{,}t+\varepsilon-\varepsilon_{,}) \qquad [1]$$

$$+ m_{,}\Big\{-\frac{b_{1,2}}{a_{,}}+\frac{a\,a_{,}}{2\,a_{,}^{3}}\Big\{\sin^2\frac{\imath_{,}}{2}+\frac{e^2+e_{,}^{2}}{2}\Big\}\Big\{b_{3,1}+b_{3,3}\Big\}+\frac{3\,(a^2e^2+a_{,}^{2}e_{,}^{2})}{2\cdot2a_{,}^{3}}b_{3,2}$$

$$+\frac{1\cdot3\cdot3}{2\cdot4}\frac{a^2a_{,}^{2}(e^2+e_{,}^{2})}{a_{,}^{5}}b_{5,0}+\frac{1\cdot3\cdot2}{2\cdot4}\frac{(a^2e^2+a_{,}^{2}e_{,}^{2})}{a_{,}^{5}}aa_{,}b_{5,1}$$

* The numbers at the side serve to distinguish the arguments.

$$-\frac{1.3}{2.4a_1^5}\left\{2a^4e^2+5a^2a_1^2(e^2+e_1^2)+2a_1^4e_1^2\right\}b_{5,2}$$

$$+\frac{1.3.2}{2.4a_1^5}(a^2e^2+a_1^2e_1^2)b_{5,3}\Big\}\cos(2nt-2n_1t+2\varepsilon-2\varepsilon_1) \quad [2]$$

$$+m_1\Big\{-\frac{b_{1,3}}{a_1}+\frac{aa_1}{2a_1^3}\Big\{\sin^2\frac{i_1}{2}+\frac{e^2+e_1^2}{2}\Big\}b_{3,2}+\frac{3(a^2e^2+a_1^2e_1^2)}{2.2a_1^3}b_{3,3}$$

$$+\frac{1.3.3}{2.4.2}\frac{a^2a_1^2(e^2+e_1^2)}{a_1^5}b_{5,1}+\frac{1.3.2}{2.4}\frac{(a^2e^2+a_1^2e_1^2)}{4a_1^5}aa_1b_{5,2}$$

$$-\frac{1.3}{2.4}\left\{2a^4e^2+5a^2a_1^2(e^2+e_1^2)+2a_1^4e_1^2)\right\}b_{5,3}\Big\}\cos(3nt-3n_1t+3\varepsilon-3\varepsilon_1) \quad [3]$$

$$+m_1\Big\{-\frac{b_{1,4}}{a_1}+\frac{aa_1}{2a_1^3}\Big\{\sin^2\frac{i_1}{2}+\frac{e^2+e_1^2}{2}\Big\}b_{3,3}+\frac{1.3.3}{2.4.2}\frac{a^2a_1^2(e^2+e_1^2)}{a_1^5}b_{5,2}$$

$$+\frac{1.3.2}{2.4}\frac{(a^2e^2+a_1^2e_1^2)}{a_1^5}aa_1b_{5,3}\Big\}\cos(4nt-4n_1t+4\varepsilon-4\varepsilon_1) \quad [4]$$

$$+m_1\Big\{-\frac{b_{1,5}}{a_1}+\frac{1.3.3}{2.4.2}\frac{a^2a_1^2(e^2+e_1^2)}{a_1^5}b_{5,3}\Big\}\cos(5nt-5n_1t+5\varepsilon-5\varepsilon_1) \quad [5]$$

$$+m_1\Big\{-\frac{3a}{2a_1^2}+\frac{3aa'}{2a_1^3}b_{3,0}-\frac{a^2}{2a_1^3}b_{3,1}-\frac{aa_1}{2.2a_1^3}b_{3,2}\Big\}e\cos(n_1t+\varepsilon_1-\varpi) \quad [6]$$

$$+m_1\Big\{-\frac{a^2}{a_1^3}b_{3,0}+\frac{aa_1}{2a_1^3}b_{3,1}\Big\}e\cos(nt+\varepsilon-\varpi) \quad [7]$$

$$+m_1\Big\{\frac{1}{2}\frac{a}{a_1^2}-\frac{aa_1}{2a_1^3}b_{3,0}-\frac{a^2}{2a_1^3}b_{3,1}+\frac{3}{2.2}\frac{aa_1}{a_1^3}b_{3,2}\Big\}e\cos(2nt-n_1t+2\varepsilon-\varepsilon_1-\varpi) \quad [8]$$

$$+m_1\Big\{-\frac{aa_1}{2.2a_1^3}b_{3,1}-\frac{a^2}{2a_1^3}b_{3,2}+\frac{3}{2.2}\frac{aa_1}{a_1^3}b_{3,3}\Big\}e\cos(3nt-2n_1t+3\varepsilon-2\varepsilon_1-\varpi) \quad [9]$$

$$+m_1\Big\{-\frac{aa_1}{2.2a_1^3}b_{3,2}-\frac{a^2}{2a_1^3}b_{3,3}\Big\}e\cos(4nt-3n_1t+4\varepsilon-3\varepsilon_1-\varpi) \quad [10]$$

$$-m_1\frac{aa_1}{2.2a_1^3}b_{3,3}\,e\cos(5nt-4n_1t+5\varepsilon-4\varepsilon_1-\varpi) \quad [11]$$

$$+m_1\Big\{\frac{3}{2.2}\frac{aa_1}{a_1^3}b_{3,1}-\frac{a^2}{2a_1^3}b_{3,2}-\frac{aa_1}{2.2a_1^3}b_{3,3}\Big\}e\cos(nt-2n_1t+\varepsilon-2\varepsilon_1+\varpi) \quad [12]$$

$$+m_1\Big\{\frac{3}{2.2}\frac{aa_1}{a_1^3}b_{3,2}-\frac{a^2}{2a_1^3}b_{3,3}\Big\}e\cos(2nt-3n_1t+2\varepsilon-3\varepsilon_1+\varpi) \quad [13]$$

$$+m_1\frac{3aa_1}{2.2a_1^3}b_{3,3}\,e\cos(3nt-4n_1t+3\varepsilon-4\varepsilon_1+\varpi) \quad [14]$$

$$+ m_{,}\left\{ - \frac{a_{,}^{2}}{a_{,}^{3}} b_{3,0} - \frac{a\,a_{,}}{2\,a_{,}^{3}} b_{3,1} \right\} e_{,}\cos(n_{,}t+\varepsilon_{,}-\varpi_{,}) \qquad [15]$$

$$+ m_{,}\left\{ \frac{3\,a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,1} - \frac{a_{,}^{2}}{2\,a_{,}^{3}} b_{3,2} - \frac{a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,3} \right\} e_{,}\cos(2nt-n_{,}t+2\varepsilon-\varepsilon_{,}-\varpi_{,}) \qquad [17]$$

$$+ m_{,}\left\{ \frac{3\,a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,2} - \frac{a_{,}^{2}}{2\,a_{,}^{3}} b_{3,3} \right\} e_{,}\cos(3nt-2n_{,}t+3\varepsilon-2\varepsilon_{,}-\varpi_{,}) \qquad [18]$$

$$+ m_{,}\frac{3\,a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,3}\, e_{,}\cos(4nt-3n_{,}t+4\varepsilon-3\varepsilon_{,}-\varpi_{,}) \qquad [19]$$

$$+ m_{,}\left\{ \frac{2\,a}{a_{,}^{2}} - \frac{a\,a_{,}}{2\,a_{,}^{3}} b_{3,0} - \frac{a_{,}^{2}}{2\,a_{,}^{3}} b_{3,1} + \frac{3}{2\,.\,2}\frac{a a_{,}}{a_{,}^{3}} b_{3,2} \right\} e_{,}\cos(nt-2n_{,}t+\varepsilon-2\varepsilon_{,}+\varpi_{,}) \qquad [20]$$

$$+ m_{,}\left\{ - \frac{a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,1} - \frac{a_{,}^{2}}{2\,a_{,}^{3}} b_{3,2} + \frac{3}{2\,.\,2}\frac{a a_{,}}{a_{,}^{3}} b_{3,3} \right\} e_{,}\cos(2nt-3n_{,}t+2\varepsilon-3\varepsilon_{,}+\varpi_{,}) \qquad [21]$$

$$+ m_{,}\left\{ - \frac{a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,2} - \frac{a_{,}^{2}}{2\,a_{,}^{3}} b_{3,3} \right\} e_{,}\cos(3nt-4n_{,}t+3\varepsilon-4\varepsilon_{,}+\varpi_{,}) \qquad [22]$$

$$- m_{,}\frac{a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,3}\, e_{,}\cos(4nt-5n_{,}t+4\varepsilon-5\varepsilon_{,}+\varpi_{,}) \qquad [23]$$

$$+ m_{,}\left\{ - \frac{a\,a_{,}}{2\,.\,8\,a_{,}^{3}} b_{3,1} - \frac{a^{2}}{2\,.\,4\,a_{,}^{3}} b_{3,2} - \frac{3}{2\,.\,8}\frac{a a_{,}}{a_{,}^{3}} b_{3,3} - \frac{1\,.\,3\,.\,9}{2\,.\,4\,.\,2}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}} b_{5,0} + \frac{1\,.\,3\,.\,3}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,1} \right.$$
$$\left. - \frac{1\,.\,3}{2\,.\,4\,.\,2}\frac{(2\,a^{2}-3\,a_{,}^{2})}{a_{,}^{5}} a^{2} b_{5,2} - \frac{1\,.\,3}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,3} \right\} e^{2}\cos(2n_{,}t+2\varepsilon_{,}-2\varpi) \qquad [24]$$

$$+ m_{,}\left\{ \frac{a}{8\,a_{,}^{2}} - \frac{a a_{,}}{2\,.\,4\,a_{,}^{3}} b_{3,0} - \frac{a^{2}}{2\,.\,4\,a_{,}^{3}} b_{3,1} - \frac{3\,a\,a_{,}}{2\,.\,8\,a_{,}^{3}} b_{3,2} + \frac{1\,.\,3\,.\,6}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,0} \right.$$
$$- \frac{1\,.\,3\,a^{2}}{2\,.\,4\,.\,4}\frac{(4\,a^{3}+3\,a_{,}^{2})}{a_{,}^{5}} b_{5,1} - \frac{1\,.\,3}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,2}$$
$$\left. - \frac{1\,.\,3}{2\,.\,4\,.\,4}\frac{a^{3}a_{,}^{2}}{a_{,}^{5}} b_{5,3} \right\} e^{2}\cos(nt+n_{,}t+\varepsilon+\varepsilon_{,}-2\varpi) \qquad [25]$$

$$+ m_{,}\left\{ - \frac{a^{2}}{2\,.\,2\,a_{,}^{3}} b_{3,0} - \frac{a\,a_{,}}{2\,.\,2\,a_{,}^{3}} b_{3,1} - \frac{1\,.\,3}{2\,.\,4}\frac{a^{2}(2\,a^{2}-3\,a_{,}^{2})}{a_{,}^{5}} b_{5,0} \right.$$
$$\left. + \frac{1\,.\,3\,.\,2}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,1} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,2}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}} b_{5,2} \right\} e^{2}\cos(2nt+2\varepsilon-2\varpi) \qquad [26]$$

$$+ m_{,}\left\{ \frac{3}{8}\frac{a}{a_{,}^{2}} - \frac{3}{8}\frac{a a_{,}}{a_{,}^{3}} b_{3,0} - \frac{a a_{,}}{2\,.\,4\,a_{,}^{3}} b_{3,1} - \frac{a^{2}}{2\,.\,8\,a_{,}^{3}} b_{3,2} - \frac{1\,.\,3\,.\,2}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,0} \right.$$
$$\left. - \frac{1\,.\,3\,a^{2}}{2\,.\,4\,.\,4}\frac{(4\,a^{3}-5\,a_{,}^{2})}{a_{,}^{5}} b_{5,1} + \frac{1\,.\,3\,.\,3}{2\,.\,4}\frac{a^{3}a_{,}}{a_{,}^{5}} b_{5,2} \right.$$

$$- \frac{1.3.9}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,3}\Big\} e^2 \cos(3nt - n_1 t + 3\varepsilon - \varepsilon_1 - 2\varpi) \quad [27]$$

$$+ m_1\Big\{ - \frac{3\,a\,a_1}{2.8\,a_1^3} b_{3,1} - \frac{a^2}{2.4\,a_1^3} b_{3,2} - \frac{a\,a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3\,a^2 a_1^2}{2.4.2\,a_1^5} b_{5,0}$$
$$- \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,1} - \frac{1.3\,a^2}{2.4.2}\frac{(2a^2 - 3a_1^2)}{a_1^5} b_{5,2}$$
$$+ \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(4nt - 2n_1 t + 4\varepsilon - 2\varepsilon_1 - 2\varpi) \quad [28]$$

$$+ m_1\Big\{ - \frac{3\,a\,a_1}{2.8\,a_1^3} b_{3,2} - \frac{a^2}{2.4\,a_1^3} b_{3,3} - \frac{1.3}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,1} - \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,2}$$
$$- \frac{1.3\,a^2}{2.4.2}\frac{(2a^2 - 3a_1^2)}{a_1^5} b_{5,3}\Big\} e^2 \cos(5nt - 3n_1 t + 5\varepsilon - 3\varepsilon_1 - 2\varpi) \quad [29]$$

$$+ m_1\Big\{ - \frac{3\,a a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,2}$$
$$- \frac{1.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(6nt - 4n_1 t + 6\varepsilon - 4\varepsilon_1 - 2\varpi) \quad [30]$$

$$- m_1 \frac{1.3}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,3}\, e^2 \cos(7nt - 5n_1 t + 7\varepsilon - 5\varepsilon_1 - 2\varpi) \quad [31]$$

$$+ m_1\Big\{ - \frac{a a_1}{2.8\,a_1^3} b_{3,2} - \frac{a^2}{2.4\,a_1^3} b_{3,3} - \frac{1.3.9}{2.4}\frac{a^2 a_1^2}{a_1^5} b_{5,1} + \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,2}$$
$$- \frac{1.3}{2.4}\frac{a^2}{2}\frac{(2a^2 - 3a_1^2)}{a_1^5} b_{5,3}\Big\} e^2 \cos(nt - 3n_1 t + \varepsilon - 3\varepsilon_1 + 2\varpi) \quad [32]$$

$$- m_1\Big\{ - \frac{a a_1}{2.8\,a_1^3} b_{3,3} - \frac{1.3.9}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,2}$$
$$+ \frac{1.3.3}{2.4}\frac{a^3 a_1}{a_1^5} b_{5,3}\Big\} e^2 \cos(2nt - 4n_1 t + 2\varepsilon - 4\varepsilon_1 + 2\varpi) \quad [33]$$

$$- m_1 \frac{1.3.9}{2.4.4}\frac{a^2 a_1^2}{a_1^5} b_{5,3}\, e^2 \cos(3nt - 5n_1 t + 3\varepsilon - 5\varepsilon_1 + 2\varpi) \quad [34]$$

$$+ m_1\Big\{ - \frac{5}{2.2}\frac{a a_1}{a_1^3} b_{3,1} + \frac{1.3.2}{2.4}\frac{a^2 a_1^2}{a_1^5} b_{5,0} + \frac{1.3.2}{2.4}\frac{(a^2 + a_1^2)}{a_1^5} a a_1 b_{5,1}$$
$$- \frac{1.3.5}{2.4}\frac{a^2 a_1^2}{a_1^5} b_{5,2}\Big\} e e_1 \cos(nt - n_1 t + \varepsilon - \varepsilon_1 - \varpi + \varpi_1) \quad [35]$$

$$+ m_1\Big\{ \frac{a}{a_1} - \frac{a a_1}{2.2\,a_1^3} b_{3,0} - \frac{9}{2.4}\frac{a a_1}{a_1^3} b_{3,2} - \frac{1.3.2}{2.4}\frac{(a^2 + a_1^2)}{a_1^5} a a_1 b_{5,0}$$

$$+\frac{1.3}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,1}+\frac{1.3.3}{2.4}\frac{a^2+a_{,}^2}{a_{,}^5}aa_{,}b_{5,2}$$

$$-\frac{1.3.9}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,3}\Big\}ee_{,}\cos(2nt-2n_{,}t+2\varepsilon-2\varepsilon_{,}-\varpi+\varpi_{,}) \quad [36]$$

$$+m_{,}\Big\{-\frac{aa_{,}}{2.4a_{,}^3}b_{3,1}-\frac{9}{2.4}\frac{aa_{,}}{a_{,}^3}b_{3,3}-\frac{1.3}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,0}-\frac{1.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,1}+\frac{1.3}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,2}$$

$$+\frac{1.3.3}{2.4}\frac{a^2+a_{,}^2}{a_{,}^5}aa_{,}b_{5,3}\Big\}ee_{,}\cos(3nt-3n_{,}t+3\varepsilon-3\varepsilon_{,}-\varpi+\varpi_{,}) \quad [37]$$

$$+m_{,}\Big\{-\frac{aa_{,}}{2.4a_{,}^3}b_{3,2}-\frac{1.3}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,1}-\frac{1.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,2}$$

$$+\frac{1.3}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,3}\Big\}ee_{,}\cos(4nt-4n_{,}t+4\varepsilon-4\varepsilon_{,}-\varpi+\varpi_{,}) \quad [38]$$

$$+m_{,}\Big\{-\frac{aa_{,}}{2.4a_{,}^3}b_{3,3}-\frac{1.3}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,2}$$

$$-\frac{1.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,3}\Big\}ee_{,}\cos(5nt-5n_{,}t+5\varepsilon-5\varepsilon_{,}-\varpi+\varpi_{,}) \quad [39]$$

$$-m_{,}\frac{1.3}{2.4}\frac{a^2a_{,}^2}{2a_{,}^5}b_{5,3}\cos(6nt-6n_{,}t-\varpi+\varpi_{,}) \quad [40]$$

$$+m_{,}\Big\{-\frac{9}{2.2}\frac{aa_{,}}{a_{,}^3}b_{3,0}-\frac{aa_{,}}{2.4a_{,}^3}b_{3,2}+\frac{1.3.6}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,0}-\frac{1.3.7}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,1}$$

$$-\frac{1.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}a_{,}ab_{5,2}-\frac{1.3}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,3}\Big\}ee_{,}\cos(\varpi-\varpi_{,}) \quad [41]$$

$$+m_{,}\Big\{-\frac{9}{2.4}\frac{aa_{,}}{a_{,}^3}b_{3,1}-\frac{aa_{,}}{2.4a_{,}^3}b_{3,3}-\frac{1.3.9}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,0}+\frac{1.3.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,1}$$

$$+\frac{1.3}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,2}-\frac{1.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,3}\Big\}ee_{,}\cos(nt-n_{,}t+\varepsilon-\varepsilon_{,}+\varpi-\varpi_{,}) \quad [42]$$

$$+m_{,}\Big\{-\frac{9}{2.4}\frac{aa_{,}}{a_{,}^3}b_{3,2}-\frac{1.3.9}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,1}+\frac{1.3.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,2}$$

$$+\frac{1.3}{2.4}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,3}\Big\}ee_{,}\cos(2nt-2n_{,}t+2\varepsilon-2\varepsilon_{,}+\varpi-\varpi_{,}) \quad [43]$$

$$+m_{,}\Big\{-\frac{9}{2.4}\frac{aa_{,}}{a_{,}^3}b_{3,3}-\frac{1.3.9}{2.4.2}\frac{a^2a_{,}^2}{a_{,}^5}b_{5,2}$$

$$+\frac{1.3.3}{2.4}\frac{(a^2+a_{,}^2)}{a_{,}^5}aa_{,}b_{5,3}\Big\}ee_{,}\cos(3nt-3n_{,}t+3\varepsilon-3\varepsilon_{,}+\varpi-\varpi_{,}) \quad [44]$$

$$- m_{,} \frac{1.3.9}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,3}\, ee_{,} \cos(4nt - 4n_{,}t + 4\varepsilon - 4\varepsilon_{,} + \varpi - \varpi_{,}) \qquad [45]$$

$$+ m_{,} \left\{ -\frac{3a}{a_{,}^2} + \frac{3 a a_{,}}{2.2 a_{,}^3} b_{3,0} + \frac{3 a a_{,}}{2.4 a_{,}^3} b_{3,2} + \frac{1.3.2}{1.4} \frac{(3a_{,}^2 - a^2)}{a_{,}^5} a a_{,} b_{5,0} - \frac{1.3.11}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,1} \right.$$

$$\left. + \frac{1.3}{2.4} \frac{(3a_{,}^2 - a^2)}{a_{,}^5} a a_{,} b_{5,2} + \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,3} \right\} ee_{,} \cos(2n_{,}t + 2\varepsilon_{,} - \varpi - \varpi_{,}) \qquad [46]$$

$$+ m_{,} \left\{ \frac{3}{2.2} \frac{a a_{,}}{a_{,}^3} b_{3,1} - \frac{1.3.14}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,0} + \frac{1.3.2}{2.4} \frac{(a^2 + a_{,}^2)}{a_{,}^5} a a_{,} b_{5,1} \right.$$

$$\left. + \frac{1.3.3}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,2} \right\} ee_{,} \cos(nt + n_{,}t + \varepsilon + \varepsilon_{,} - \varpi - \varpi_{,}) \qquad [47]$$

$$+ m_{,} \left\{ \frac{3 a a_{,}}{2.2 a_{,}^3} b_{3,0} + \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,2} + \frac{1.3.2}{2.4} \frac{(3a^2 - a_{,}^2)}{a_{,}^5} a a_{,} b_{5,0} - \frac{1.3.11}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,1} \right.$$

$$\left. + \frac{1.3}{2.4} \frac{(3a_{,}^2 - a^2)}{a_{,}^5} a a_{,} b_{5,2} + \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,3} \right\} ee_{,} \cos(2nt + 2\varepsilon - \varpi - \varpi_{,}) \qquad [48]$$

$$+ m_{,} \left\{ \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,1} + \frac{3 . a a_{,}}{2.4 a_{,}^3} b_{3,3} + \frac{1.3.3}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,0} \right.$$

$$+ \frac{1.3}{2.4} \frac{(3a^2 - a_{,}^2)}{a_{,}^5} a a_{,} b_{5,1} - \frac{1.3.7}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,2}$$

$$\left. + \frac{1.3}{2.4} \frac{(3a_{,}^2 - a^2)}{a_{,}^5} a a_{,} b_{5,3} \right\} ee_{,} \cos(3nt - n_{,}t + 3\varepsilon - \varepsilon_{,} - \varpi - \varpi_{,}) \qquad [49]$$

$$+ m_{,} \left\{ \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,2} + \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,1} + \frac{1.3}{2.4} \frac{(3a^2 - a_{,}^2)}{a_{,}^5} a a_{,} b_{5,2} \right.$$

$$\left. - \frac{1.3.7}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,3} \right\} ee_{,} \cos(4nt - 2n_{,}t + 4\varepsilon - 2\varepsilon_{,} - \varpi - \varpi_{,}) \qquad [50]$$

$$+ m_{,} \left\{ \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,3} + \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} a a_{,} b_{5,2} \right.$$

$$\left. + \frac{1.3}{2.4} \frac{(3a^2 - a_{,}^2)}{a_{,}^5} a a_{,} b_{5,3} \right\} ee_{,} \cos(5nt - 3n_{,}t + 5\varepsilon - 3\varepsilon_{,} - \varpi - \varpi_{,}) \qquad [51]$$

$$+ m_{,} \frac{1.3.3}{2.4.2} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,3}\, ee_{,} \cos(6nt - 4n_{,}t + 6\varepsilon - 4\varepsilon_{,} - \varpi - \varpi_{,}) \qquad [52]$$

$$+ m_{,} \left\{ \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,1} + \frac{3}{2.4} \frac{a a_{,}}{a_{,}^3} b_{3,3} + \frac{1.3.3}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,0} \right.$$

$$\left. + \frac{1.3}{2.4} \frac{(3a_{,}^2 - a^2)}{a_{,}^5} a a_{,} b_{5,1} - \frac{1.3.7}{2.4} \frac{a^2 a_{,}^2}{a_{,}^5} b_{5,2} \right.$$

$$+\frac{1.3}{2.4}\frac{(3a^2-a_l^2)}{a_l^5}aa_l b_{5,3}\Big\}\, ee_l\cos(nt-3n_lt+\varepsilon-3\varepsilon_l+\varpi+\varpi_l) \qquad [53]$$

$$+m_l\Big\{\frac{3}{2.4}\frac{a\,a_l}{a_l^3}b_{3,1}+\frac{3}{2.4}\frac{a\,a_l}{a_l^3}b_{3,2}+\frac{1.3.3}{2.4}\frac{a^2a_l^2}{a_l^5}b_{5,0}$$

$$+\frac{1.3}{2.4}\frac{(3a_l^2-a^2)}{a_l^5}aa_l b_{5,1}+\frac{1.3}{2.4}\frac{(3a^2-a_l^2)}{a_l^5}aa_l b_{5,2}$$

$$-\frac{1.3.7}{2.4}\frac{a^2a_l^2}{a_l^5}b_{5,3}\Big\}\, ee_l\cos(2nt-4n_lt+2\varepsilon-4\varepsilon_l+\varpi+\varpi_l) \qquad [54]$$

$$+m_l\Big\{\frac{3}{2.4}\frac{a\,a_l}{a_l^3}b_{3,3}+\frac{1.3.3}{2.4.2}\frac{a^2a_l^2}{a_l^5}b_{5,2}$$

$$+\frac{1.3}{2.4}\frac{(3a_l^2-a^2)}{a_l^5}aa_l b_{5,3}\Big\}\, ee_l\cos(3nt-5n_lt+3\varepsilon-5\varepsilon_l+\varpi+\varpi_l) \qquad [55]$$

$$+m_l\frac{1.3.3}{2.4.2}\frac{a^2a_l^2}{a_l^5}b_{5,3}\,ee_l\cos(4nt-6n_lt+4\varepsilon-6\varepsilon_l+\varpi+\varpi_l) \qquad [56]$$

$$+m_l\Big\{-\frac{a_l^2}{2.2\,a_l^3}b_{3,0}-\frac{a\,a_l}{2.2\,a_l^3}b_{3,1}-\frac{1.3}{2.4}\frac{(2a_l^2-3a^2)}{a_l^5}a_l^2b_{5,0}$$

$$+\frac{1.3.2}{2.4}\frac{a\,a_l^3}{a_l^5}b_{5,1}-\frac{1.3.5}{2.4.2}\frac{a^2a_l^2}{a_l^5}b_{5,2}\Big\}\, e_l^2\cos(2n_lt+2\varepsilon_l-2\varpi_l) \qquad [57]$$

$$+m_l\Big\{\frac{a}{8\,a_l^2}-\frac{a\,a_l}{2.4\,a_l^3}b_{3,0}-\frac{a_l^2}{2.4\,a_l^3}b_{3,1}-\frac{3\,a\,a_l}{2.8\,a_l^3}b_{3,2}+\frac{1.3.6}{2.4}\frac{a_l^3a}{a_l^5}b_{5,0}$$

$$-\frac{1.3}{2.4.4}\frac{(4a^2+3a_l^2)}{a_l^5}a_l^2b_{5,1}-\frac{1.3}{2.4}\frac{a\,a_l^3}{a_l^5}b_{5,2}$$

$$-\frac{1.3}{2.4.4}\frac{a^2a_l^2}{a_l^5}b_{5,3}\Big\}\, e_l^2\cos(nt+n_lt+\varepsilon+\varepsilon_l-2\varpi_l) \qquad [58]$$

$$+m_l\Big\{-\frac{a\,a_l}{2.8\,a_l^3}b_{3,1}-\frac{a_l^2}{2.4\,a_l^3}b_{3,2}-\frac{3}{2.8}\frac{aa_l}{a_l^3}b_{3,3}-\frac{1.3.9}{2.4.2}\frac{a^2a_l^2}{a_l^5}b_{5,0}+\frac{1.3.3}{2.4}\frac{a\,a_l^3}{a_l^5}b_{5,1}$$

$$-\frac{1.3}{2.4.2}\frac{(2a_l^2-3a^2)}{a_l^5}a_l^2b_{5,2}-\frac{1.3}{2.4}\frac{a\,a_l^3}{a_l^5}b_{5,3}\Big\}\, e_l^2\cos(2nt+2\varepsilon-2\varpi_l) \qquad [59]$$

$$+m_l\Big\{-\frac{a\,a_l}{2.8\,a_l^3}b_{3,2}-\frac{a_l^2}{2.4\,a_l^3}b_{3,3}-\frac{1.3.9}{2.4.4}\frac{a^2a_l^2}{a_l^5}b_{5,1}+\frac{1.3.3}{2.4}\frac{a_l^3a}{a_l^5}b_{5,2}$$

$$-\frac{1.3}{2.4.2}\frac{(2a_l^2-3a^2)}{a_l^5}a_l^2b_{5,3}\Big\}\, e_l^2\cos(3nt-n_lt+3\varepsilon-\varepsilon_l-2\varpi_l) \qquad [60]$$

$$+m_l\Big\{-\frac{a\,a_l}{2.8\,a_l^3}b_{3,3}-\frac{1.3.9}{2.4.4}\frac{a^2a_l^2}{a_l^5}b_{5,2}$$

$$+\frac{1.3.3}{2.4}\frac{a_{,}^{3}a}{a_{,}^{5}}b_{5,3}\Big\}e_{,}^{2}\cos(4nt-2n_{,}t+4\varepsilon-2\varepsilon_{,}-2\varpi_{,}) \quad [61]$$

$$-m_{,}\frac{1.3.9}{2.4.4}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,3}e_{,}^{2}\cos(5nt-3n_{,}t+5\varepsilon-3\varepsilon_{,}-2\varpi_{,}) \quad [62]$$

$$+m_{,}\Big\{\frac{27}{8}\frac{a}{a_{,}^{2}}-\frac{3aa_{,}}{2.4a_{,}^{3}}b_{3,0}-\frac{a_{,}^{2}}{2.4a_{,}^{3}}b_{3,1}-\frac{aa_{,}}{2.8a_{,}^{3}}b_{3,2}-\frac{1.3.2}{2.4}\frac{a_{,}^{3}a}{a_{,}^{5}}b_{5,0}$$
$$-\frac{1.3}{2.4.4}\frac{(4a_{,}^{2}-5a^{2})}{a_{,}^{5}}a_{,}^{2}b_{5,1}+\frac{1.3.3}{2.4}\frac{aa_{,}^{3}}{a_{,}^{5}}b_{5,2}$$
$$-\frac{1.3.9}{2.4.4}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,3}\Big\}e_{,}^{2}\cos(nt-3n_{,}t+\varepsilon-3\varepsilon_{,}+2\varpi_{,}) \quad [63]$$

$$+m_{,}\Big\{-\frac{3aa_{,}}{2.8a_{,}^{3}}b_{3,1}-\frac{a_{,}^{2}}{2.4a_{,}^{3}}b_{3,2}-\frac{aa_{,}}{2.8a_{,}^{3}}b_{3,3}-\frac{1.3}{2.4.2}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,0}$$
$$-\frac{1.3}{2.4}\frac{aa_{,}^{3}}{a_{,}^{5}}b_{5,1}-\frac{1.3}{2.4.2}\frac{(2a_{,}^{2}-3a^{2})}{a_{,}^{5}}a_{,}^{2}b_{5,2}$$
$$+\frac{1.3.3}{2.4}\frac{aa_{,}^{3}}{a_{,}^{5}}b_{5,3}\Big\}e_{,}^{2}\cos(2nt-4n_{,}t+2\varepsilon-4\varepsilon_{,}+2\varpi_{,}) \quad [64]$$

$$+m_{,}\Big\{-\frac{3aa_{,}}{2.8a_{,}^{3}}b_{3,2}-\frac{a_{,}^{2}}{2.4a_{,}^{3}}b_{3,3}-\frac{1.3}{2.4.4}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,1}-\frac{1.3}{2.4}\frac{aa_{,}^{3}}{a_{,}^{5}}b_{5,2}$$
$$-\frac{1.3}{2.4.2}\frac{(2a_{,}^{2}-3a^{2})}{a_{,}^{5}}a_{,}^{2}b_{5,3}\Big\}e_{,}^{2}\cos(3nt-5n_{,}t+3\varepsilon-3\varepsilon_{,}+2\varpi_{,}) \quad [65]$$

$$+m_{,}\Big\{-\frac{3}{2.8}\frac{aa_{,}}{a_{,}^{3}}b_{3,3}-\frac{1.3}{2.4.4}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,2}$$
$$-\frac{1.3}{2.4}\frac{aa_{,}^{3}}{a_{,}^{5}}b_{5,3}\Big\}e_{,}^{2}\cos(4nt-6n_{,}t+4\varepsilon-6\varepsilon_{,}+2\varpi_{,}) \quad [66]$$

$$-m_{,}\frac{1.3}{2.4.4}\frac{a^{2}a_{,}^{2}}{a_{,}^{5}}b_{5,3}e_{,}^{2}\cos(5nt-7n_{,}t+5\varepsilon-7\varepsilon_{,}+2\varpi_{,}) \quad [67]$$

$$+m_{,}\Big\{\frac{a}{a_{,}^{2}}\Big(1+\frac{3}{2}e_{,}^{2}\Big)-\frac{aa_{,}}{a_{,}^{3}}b_{3,0}\Big\}\sin^{2}\frac{\iota_{,}}{2}\cos(nt+n_{,}t+\varepsilon+\varepsilon_{,}-2\nu_{,}) \quad [68]$$

$$-m_{,}\frac{aa_{,}}{2a_{,}^{3}}b_{3,1}\sin^{2}\frac{\iota_{,}}{2}\cos(2n_{,}t+2\varepsilon_{,}-2\nu_{,}) \quad [69]$$

$$m\frac{aa_{,}}{2a_{,}^{3}}b_{3,1}\sin^{2}\frac{\iota_{,}}{2}\cos(2nt+2\varepsilon-2\nu_{,}) \quad [70]$$

$$-m_{,}\frac{aa_{,}}{2a_{,}^{3}}b_{3,2}\sin^{2}\frac{\iota_{,}}{2}\cos(nt+3n_{,}t+\varepsilon-3\varepsilon_{,}+2\nu_{,}) \quad [71]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{3}}b_{3,2}\sin^{2}\tfrac{\iota_{,}}{2}\cos(3nt-n_{,}t+3\varepsilon-\varepsilon_{,}-2\nu_{,}) \qquad [72]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{3}}b_{3,3}\sin^{2}\tfrac{\iota_{,}}{2}\cos(2nt-4n_{,}t+2\varepsilon-4\varepsilon_{,}+2\nu_{,}) \qquad [73]$$

$$-m_{,}\frac{a\,a_{,}}{2\,a_{,}^{3}}b_{3,3}\sin^{2}\tfrac{\iota_{,}}{2}\cos(4nt-2n_{,}t+4\varepsilon-2\varepsilon_{,}-2\nu_{,}) \qquad [74]$$

In the development of R, I have supposed $\iota = 0$, so that $\iota_{,}$ is the angle contained between the orbits of the planets P and $P_{,}$, or P_1 and P_2; in the general case, when ι_1, and ι_2, are retained, $\cos\iota_{,} = \cos\iota_1\cos\iota_2 + \sin\iota_1\sin\iota_2\cos(\iota_1 - \iota_2)$, ι_1 and ι_2, being the inclinations of the orbits of the planets P_1, P_2 to any plane $x\,y$, of which the direction is arbitrary,

$$r'r_{,}'\sin(\lambda' - \lambda_{,}') = r_1 r_2\Big\{\cos^2\tfrac{\iota_1}{2}\cos^2\tfrac{\iota_2}{2}\sin(\lambda_1 - \lambda_2)$$

$$-\sin^2\tfrac{\iota_1}{2}\cos^2\tfrac{\iota_2}{2}\sin(\lambda_1 + \lambda_2 - 2\nu_1) + \sin^2\tfrac{\iota_2}{2}\cos^2\tfrac{\iota_1}{2}\sin(\lambda_1 + \lambda_2 - 2\nu_2)$$

$$-\sin^2\tfrac{\iota_1}{2}\cos^2\tfrac{\iota_2}{2}\sin(\lambda_1 - \lambda_2 - 2\nu_1 + 2\nu_2)\Big\}$$

$$r'r_{,}'\Big\{\cos(\lambda' - \lambda_{,}') + s\,s_{,}\Big\} = r_1 r_2\Big\{\cos^2\tfrac{\iota_1}{2}\cos^2\tfrac{\iota_2}{2}\cos(\lambda_1 - \lambda_2)$$

$$+\sin^2\tfrac{\iota_1}{2}\cos^2\tfrac{\iota_2}{2}\cos(\lambda_1 + \lambda_2 - 2\nu_1) + \sin^2\tfrac{\iota_2}{2}\cos^2\tfrac{\iota_1}{2}\cos(\lambda_1 + \lambda_2 - 2\nu_2)$$

$$+\sin^2\tfrac{\iota_1}{2}\sin^2\tfrac{\iota_2}{2}\cos(\lambda_1 - \lambda_2 - 2\nu_1 + 2\nu_2)$$

$$+\sin\iota_1\sin\iota_2\cos(\lambda_1 - \lambda_2 - \nu_1 + \nu_2) - \sin\iota_1\sin\iota_2\cos(\lambda_1 + \lambda_2 - \nu_1 - \nu_2)\Big\}$$

Errata.

In page 330, *for* ι and ν, *read* $\iota_{,}$ and $\nu_{,}$.